（別紙様式５）

変革を駆動する先端物理・数学プログラム（FoPM）

国外連携機関長期研修　報告書

氏　　名	Renji Sakamoto
所属部局	数理科学研究科　　数理科学専攻
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My research concerns the structure of maximal commutative subsemigroups in the semigroup of tropical matrices. Although these objects arise naturally in tropical algebra, their geometric structure is not yet fully understood. In particular, I have been interested in whether they can be studied not only as algebraic objects but also from broader tropical-geometric perspectives. Since Professor Diane Maclagan is one of the leading researchers in tropical geometry, I considered it highly meaningful to discuss my project with her and with researchers at Warwick, where active interaction across related areas is possible.

The duration of my visit was three weeks, during which Professor Maclagan and I discussed the structures of maximal commutative subsemigroups in the semigroup of tropical matrices. We held several meetings each week and had many fruitful discussions throughout my stay.

In the first week, I presented my current research to Professor Maclagan and her group members, received feedback from them, and discussed several related questions. For example, some members pointed out connections between our objects and other perspectives in tropical geometry, such as tropical convexity. This was important because it suggested that maximal commutative subsemigroups should not be treated as isolated algebraic objects, but rather as objects that may admit a richer geometric interpretation. In particular, considering them in relation to tropical convexity raised the possibility that parts of their structure might be described using concepts that are already well established in tropical geometry. This changed the direction of our subsequent discussions, since we began to ask not only what these subsemigroups are algebraically, but also how they should be situated within a broader geometric framework.

In the second week, we pursued this topic further. The institute was equipped with blackboards almost everywhere, and we held discussions in various places throughout the department. This made it possible to involve other researchers in our conversations and develop them further. Based on the discussions in the previous week, Professor Maclagan pointed out that the geometric structures of maximal commutative subsemigroups resemble typical tropical varieties, although they are not identical to them. This observation led us to the notion of realization spaces. The idea was that, if our objects can be regarded as tropical shadows of more classical spaces over a valuated field, then one may hope to recover their structure by studying which lifts are possible and how such lifts vary. In this way, the problem is transformed from a purely tropical one into a question about the relationship between tropical data and classical geometric realizations. Accordingly, during this week, Professor Maclagan and I mainly discussed the possibility of constructing realization spaces for maximal commutative subsemigroups. In other words, we considered lifts of points in our objects to affine spaces over a field with non-trivial valuation via tropicalization, and we observed aspects of how these lifts behave with respect to multiplication, as well as indications of hidden geometry in the realization spaces. We found that some lifts and their products correspond to tropical matrices and their products. On the other hand, it also became clear that additional points must be taken into account in order to generate the whole space.

In the final week, we focused on the extent to which this geometric viewpoint can be expected to work consistently. In particular, we discussed several sources of irregularity that seem to arise when one attempts to relate tropical structures to their counterparts over valued fields. This part of the discussion was important because it showed that the problem is not merely a straightforward translation from one setting to another, but rather involves genuinely subtle features that must be understood on their own terms. We also considered whether ideas from the theory of

buildings might provide a useful conceptual framework for organizing these phenomena. Although this line of thought is still preliminary, it opened up a promising direction by suggesting that the spaces we study may fit into a broader geometric context than we had initially assumed.

As a result of the visit, I obtained several new perspectives on the geometry of maximal commutative subsemigroups. Most importantly, we found that these objects may be studied through realization spaces over valuated fields, rather than only within tropical algebra itself. This viewpoint makes it possible to formulate the problem in a way that connects tropical data, lifts, and multiplicative structure. At the same time, our discussions clarified that the behavior of lifts is delicate and that irregularities caused by the gap between valuation and tropical operations must be taken seriously. These observations significantly refined my understanding of the problem and provided a more concrete agenda for future work.

The next step is to examine concrete families of examples more systematically and to determine under what conditions realization spaces of maximal commutative subsemigroups can be constructed in a satisfactory way. It will also be necessary to understand more precisely which lifts are compatible with multiplication and how much of the tropical structure can be recovered from them. In addition, the possible use of building theory remains a promising direction, especially for organizing the geometry that appears when lifts are taken from linear groups. I expect that these questions will play a central role in the further development of this project.

Another valuable aspect of the visit was the style of mathematical interaction at the institute. Because discussions could begin spontaneously and continue across different places within the department, ideas were tested and reformulated in an ongoing way rather than being confined to a single formal meeting. This made it easier to revisit questions raised earlier, refine tentative observations, and incorporate comments from researchers with related expertise. Such a setting was particularly beneficial for my project, since the problem required repeated comparison between algebraic intuition and geometric interpretation.

