（別紙様式５）

変革を駆動する先端物理・数学プログラム（FoPM）

国外連携機関長期研修　報告書

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受入先	University of Verona（Peter Schuster 教授）
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Overview

I visited Peter Schuster at the University of Verona for two months and conducted collaborative research on constructive algebra. I discussed several topics in constructive algebra twice a week with Peter Schuster, Ihsen Yengui, Riccardo Borsetto, Giulio Fellin, and Martin Trucchi.

I participated in the following workshops:

- Dagstuhl Seminar 26121: Proof Systems in Actual Practice: Reasoning and Computation. Wadern, March 15–20, 2026. https://www.dagstuhl.de/26121
- 7th Workshop on Formal Topology: From topology to dynamic constructivism, and back. Venice, April 13–17, 2026. https://events.math.unipd.it/7wftop/
- Taming the Infinite: Logic, Algebra, and Constructive Mathematics. Verona, April 20, 2026. https://ps60.logicverona.it

I gave the following talks:

- “Entailment relations for the constructive theory of free modules” in Verona
- “Toward constructive Birkhoff’s representation theorem” in Verona
- “Constructive theory of Jacobson rings” at Dagstuhl
- “Constructive Jacobson’s commutativity theorem” in Verona
- “Ordinals, quantitative Dickson’s lemma, and quantitative Hilbert’s basis theorem” in Verona

I participated in the following courses:

- “Advanced course in foundations of mathematics” by Peter Schuster
- “Advances in Algebra – Generalised Buchberger and Schreyer Algorithms for Coherent Rings” by Ihsen Yengui
- “Algebraic Geometry” by Georgios Dalezios and Giulio Fellin

Research with Peter Schuster

We conducted research on Jacobson’s commutativity theorem in constructive algebra. Constructive algebra is the study of algebra without non-constructive principles such as Zorn’s lemma and the law of excluded middle. Constructive proofs are important because they give an algorithm to compute mathematical objects. In constructive algebra, we often search for a method that translates a non-constructive proof into a constructive one and magically extracts an algorithm from a non-constructive proof.

Our motivation was to give a constructive proof of Jacobson’s commutativity theorem, which asserts that if for every element x of a ring A, there exists a natural number $n \geq 2$ such that $x^n = x$, then the ring A is commutative. The standard classical proof uses Zorn’s lemma to reduce the problem to the case where A is a domain. A proof which does not use Zorn’s lemma is known, but it uses the law of excluded middle.

During my visit, we obtained an important lemma on finitely enumerable algebra. By combining it with the result that I obtained in 2024, we completed the constructive proof of Jacobson’s commutativity theorem. Classically, there are plenty of generalizations of Jacobson’s commutativity theorem, and giving a constructive proof for them is left for future work.

Dagstuhl Seminar 26121

I was invited to participate in the Dagstuhl seminar. I met Martín Escardó at Frankfurt Airport, and we went to Schloss Dagstuhl together. I gave a talk on the constructive theory of Jacobson rings and shared an open question

on Jacobson rings and maximal ideals. After my talk, I had a discussion with Ingo Blechschmidt on the treatment of maximal ideals by generalized inductive definition. My research on Jacobson rings began in 2024, but it seems that there is still a lot of work to be done.

Besides the talks, several working groups were held. I learned a lot about synthetic algebraic geometry and synthetic computability theory. I also had a discussion with Thierry Coquand about my work on the quantitative version of Hilbert's basis theorem and the constructive Krull dimension.



My talk at Dagstuhl

7th Workshop on Formal Topology

The workshop was held in Venice, which is a beautiful city, as I expected. I ate a lot of delicious fish dishes, pizza, and tiramisu. I liked Blechschmidt's talk, which explains the relationship between point-free topology and constructive algebra. The relationship has been considered for a long time, and I believe it continues to be the most important technique in constructive algebra.



Venice at night

Taming the Infinite

This workshop was held to celebrate the 60th birthday of Peter Schuster. Most of the talks were quite relevant to my research area, so I enjoyed the workshop a lot. Coquand's talk on quantifier elimination was insightful for me, and I must examine the topic more.

On the day before the workshop, I looked around the city of Verona and attended a social dinner. I saw the Verona Arena, where the closing ceremony of the Milano Cortina 2026 Winter Olympics was held.



Arena di Verona

Daily life

I stayed at the university guesthouse in the Borgo Roma campus. The campus is about 15 minutes by bus from Verona Porta Nuova station, and I spent most of my time on the campus. Regarding meals, I cooked a lot for myself. I was saved by the new Lidl supermarket at Via Legnago, which opened right before I visited Verona. I often had lunch with Peter Schuster and Martin Trucchi at the university's mensa (which means cafeteria in Italian).

Acknowledgments

I would like to thank Peter Schuster for kindly accepting my visit, for insightful discussions, for providing a lot of opportunities to interact with researchers in Europe, and for supporting my life in Verona. I would also like to thank the organizers of the workshops for organizing fantastic workshops. I would like to thank Ihsen Yengui, Giulio Fellin, Riccardo Borsetto, Martin Trucchi, Ingo Blechschmidt, Iosif Petrakis, Martín Escardó, Thierry Coquand, Henri Lombardi, Stefan Neuwirth, Hajime Ishihara, Takako Nemoto, Satoru Niki, Satoshi Nakata, and all the other people I met for interesting discussions and helpful support.

I would like to thank FoPM for financial support and the University of Verona for letting me stay at the guesthouse for free.