（別紙様式５）

変革を駆動する先端物理・数学プログラム（FoPM）

国外連携機関長期研修　報告書

氏　　名	洞龍弥 (Ryuya Hora)
所属部局	数理科学研究科　数理科学専攻
受入先	Paris-Saclay University と Grothendieck Institute
日程	西暦　2025 年　4 月　10 日　～　西暦 2025 年　6 月 27 日

1. Discussion with Professor Olivia Caramello

I visited Professor Olivia Caramello in Paris, who is affiliated with the Paris-Saclay University and is the president of the Grothendieck institute, to pursue the topic of topos-theoretic automata theory. She mainly studies relative topos theory and its applications to model theory and computer science, which are closely related to my interest in the topos-theoretic approach to automata theory. We share the same core idea: making bridges between seemingly distinct fields via topoi, which has been developed under her slogan: *toposes as 'bridges.'* We had a nice discussion on the idea of connecting word combinatorics in automata theory and (Galois-theoretic) topology. We also discussed the notion of a *local state classifier*, which I introduced in my master thesis. Even though it played a central role in topos-theoretic automata theory, we did not have a fully general and practical method to calculate it, and we had to find an ad-hoc way for each topos. Olivia Caramello, who has numerous results that make abstract topos theories applicable to concrete computations, suggested some computational methods to me, and we obtained a new description of a local state classifier. This research is not yet fully done, and it will continue. We also had a discussion on more general "representation phenomena" in topos theory. Although she was extremely busy, she kindly welcomed me and had time to discuss several topics. I sincerely appreciate her support.

2. Discussion with other researchers

Besides the discussion with Olivia Caramello, I gave 6 talks in the 79-day visit, twice at Centre Lagrange (Grothendieck Institute), twice at IRIF (Université Paris Cité), once at LIPN (Sorbonne university), and once at LIX (École Polytechnique). All the talks were about topos-theoretic approach to automata theory, but depending on the audience, the focus was sometimes on topos-side and on automata-side. First 60 days (out of 79 days) were quite busy due to the preparation for those talks. I believe that these presentations increased my visibility in the field.

I would say these talks were successful in the sense that I unexpectedly had many opportunities to have follow-up discussions not only just after the talk, but also on later days. First, I talked a lot with Axel Osmond, who is a PD at Grothendieck Institute studying topos theory. We talked about the local state classifier and its potential connection with the notion of localic isotropy group of topoi. I personally have thought about this idea before without making any actual progress. However, thanks to the discussion with Axel, we were able to calculate several examples and to obtain one mathematically stated conjecture, which was unexpectedly beautiful both from conceptual and technical perspective. We have started research project on this point. We also talked about a connection between my study on complete connectedness of topoi and his joint work with Olivia Caramello on a double categorical approach to topos theory.

I also had several opportunities to talk with Morgan Rogers at LIPN (Sorbonne University), with whom I've been studying subtopoi of free monoid action topos. This topic is also closely related to topos-theoretic automata theory. We talked about both a very abstract point of view and concrete word combinatorics. An unexpected finding is a potential connection with set theory. I could have discussions with Joshua Wrigley at IRIF (Université Paris Cité) as well, with whom I have had some online discussions even before the visit, about the model-theoretic aspect of topos theory and etendue. I also had the opportunity to have discussions with Sam, Jérémie, Zeilberger, Lafforgue, Leo, Marco, Pablo, Farzad, Le, Vincent, Jonas, Quantin, José, and many other category theorists, topos theorists, theoretical computer scientists, and AI-researchers. I really appreciate all of them.

While my visit, I got one completely unexpected event, Johnstone's talk! Professor Peter Johnstone is the leading

topos-theorist, who is also known as the author of the standard topos theory textbook: *Sketches of an Elephant*. It was suddenly announced that he was going to give a talk at Oxford university about my research, a local state classifier! I found this event would be an exceptional experience for me, and I instantly decided to go to the UK. His talk was basically an explanation of my paper, but he contained many new observations and intriguing ideas. Although I do not know if I am allowed to disclose what he said (it was not recorded unfortunately), I can say that it was an extremely encouraging experience for me.

3. Life in Paris

The city of Paris was always overwhelming for me in a good way. The institute is located in the 7th arrondissement of Paris, which includes Eiffel tower, Hôtel des Invalides, and Musée d'Orsay. Even getting to the institute each day allowed me to experience the historical scenery of Paris and the calming walk along the Seine. There was a pretty awful sewage issue at my accommodation, which I would rather not describe here, but overall, life in Paris was comfortable and fulfilling thanks to kind colleagues.

Évariste Galois, who is basically at the origin of my prodiscrete monoid approach to topoi of automata, spent his last days before prison on a little street in Paris. I went to see the house myself. I also had the opportunity to visit IHÉS, where Grothendieck introduced the notion of topoi. I found and bought a French official stamp with Grothendieck's portrait on it. Being able to do research in places where Galois and Grothendieck once lived, contemplating their last days, and feeling their influence still alive in the mathematical community (not only in the mathematical contents) was something that meant a lot to me.



研究所周辺の景色