# 変革を駆動する先端物理・数学プログラム (FoPM)

# 国外連携機関長期研修 報告書

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## 1. Overview of the Visit

From March 2 to March 30, 2025, I visited the University of California, Los Angeles (UCLA) to conduct research under the supervision of Professor Igor Pak. During my stay, I had the opportunity to interact with researchers and students specializing in combinatorics and to give a presentation on my own research, through which I gained new perspectives. I also worked on a topic that Professor Pak has been interested in, the enumeration of triangulations. Through this work, I came to better understand the difficulty of the problem and explored potential new approaches.

#### 2. Research Presentation and Discussions

During the visit, I participated in the Combinatorics Seminar at UCLA and gave a presentation on my research. The talk began with a discussion on the "naturalness" of bijections and signed bijections (called *sijections*), which was the main topic of my master's thesis. I then presented a notion of compatibility and its application to the ASM-DPP bijection problem, and finally shared recent results related to lattice paths and the Lindström–Gessel–Viennot (LGV) lemma. The audience consisted of students specializing in combinatorics, and they actively engaged with my talk both during and after the presentation.

One of the most interesting points during the discussion was the question of what makes a bijection or a sijection "natural" or "constructive." Some participants suggested that this should be judged based on computational complexity—whether the bijection can be computed efficiently. In fact, it is also the opinion of Professor Pak. This was a new idea for me. At the same time, I don't completely agree. I think computational complexity is one way to evaluate a bijection, but not the only one. Even if a bijection cannot be computed in polynomial time, it can still be valuable as a mathematical construction. Also, even if such a sijection is not considered "natural" in the usual sense, it can still be meaningful if it is compatible—a concept I use in my own work. This compatibility helps preserve certain useful properties in combinatorial proofs. In the context of constructing "ordinary bijections", this kind of compatibility is considered valuable. In fact, in a graduate-level course on bijective combinatorics that I attended during my visit, Professor Pak emphasized that one of the main values of bijective proofs is that they can give proofs of refined enumerations. The idea of compatibility in sijections helps support this kind of value.

### 3. Research Activities During the Visit

The main focus of my research during the visit was the enumeration of triangulations. This is a problem that Professor Pak has long been interested in, and it became the central topic of our discussions.

In particular, Professor Pak posed the following conjecture: For every sufficiently large integer M, is there a configuration of points in the plane that has exactly M triangulations? Since there are a huge number of possible point configurations, a positive answer is expected. However, it is known that the number of triangulations grows exponentially with respect to the number of points N, so any constructive proof would need to provide exponentially many distinct configurations for each N—an extremely difficult challenge.

On the other hand, it is known that if we define A\_M as the number of integers less than or equal to M that can be realized as the number of triangulations of some point configuration, then A\_M is at least of order (log M)^2. This follows from constructing O(N) configurations with distinct numbers of triangulations for each N, the number of points in the configuration.

With this background, I shifted my focus during the latter half of the visit to evaluating upper and lower bounds on the *maximum* and *minimum* number of triangulations that can occur for *n* points in the plane. In particular, I explored the lower bounds on the *minimum* number of triangulations. Although I came up with a few ideas, I was not able to improve the known bounds. I plan to continue studying this problem and seek new approaches in the future.

# 4. Summary and Future Outlook

Through this visit, I was able to experience UCLA's active research environment and engage with researchers and students who share an interest in combinatorics. The idea of evaluating the naturalness of bijections and sijections from a complexity-theoretic perspective was especially impactful and may influence my future work.

Working on the enumeration of triangulations also gave me a deeper understanding of its complexity and allowed me to explore new research directions. Moving forward, I aim to build upon the insights gained during this visit to develop new problems and approaches, ultimately leading to more concrete results. I also hope to reconnect with the professors, postdocs, and graduate students I met during this visit at future workshops or conferences and explore the possibility of collaborative research.



My working place at UCLA