（別紙様式５）

変革を駆動する先端物理・数学プログラム（FoPM）

国外連携機関長期研修　報告書

氏　　名	WANG PEIDUO
所属部局	数理科学　研究科　　　　　数理科学　専攻
受入先	Université Grenoble Alpes, Institute Fourier
日程	西暦　2023 年　11 月　13 日　～　西暦　2023 年　12 月　12 日

Exploring the Depths of p-adic Differential Equations: A Journey at Institute Fourier

This is a significant academic journey for me to join the Institute Fourier at Université Grenoble Alpes, France, to delve deeper into the fascinating theory of exponents of p-adic differential equations. Under the expert guidance of Professor Andrea Pulita, I embarked on a path that not only broadened my understanding of this complex field but also allowed me to contribute to the ongoing research.

Master Thesis Presentation: To Talk and to Learn

A highlight of my time at the Institute was presenting my master thesis titled "On generalized Fuchs theorem over p-adic polyannuli." This presentation was an opportunity to introduce the audience to the intricate world of differential equations with regular singularities over complex numbers and more importantly, their p-adic counterparts. The latter which satisfy the Robba condition, are often viewed as a p-adic analogue of the former. My talk also included an abstract of the proof of the generalized p-adic Fuchs theorem over higher-dimensional spaces, a topic that has intrigued mathematicians for its complexity and depth.

Collaborative Research with Professor Pulita

I work with professor Pulita and a PhD student of his on topics of exponents of p-adic differential equations:

1. We discussed about detailed proof of p-adic Fuchs theorem for annuli written in Kedlaya's book p-adic differential equations, whose original idea comes from Dwork.
2. We intended to find a more intrinsic, direct and reasonable way to define exponents for p-adic differential equations over annuli, satisfying the Robba condition, using Frobenius antecedent and representation theory, which is different from Dwork's terminology.
3. We discussed application of p-adic Fuchs theorem in research about finite dimensionality of de Rham cohomology of quasi-smooth Berkovich curves.

Rethinking Dwork's Definition of Exponents

While Kedlaya's book is a cornerstone in the field, it omits many technical computational details. In our seminars, I took the initiative to write down these intricate details along with comments of professor Pulita and his student, to make the whole complex story more accessible and understandable. However, Professor Pulita pointed out a critical aspect: the proof style, reminiscent of Dwork, was heavily reliant on computations, obscuring a clear and comprehensive understanding of the p-adic Fuchs theorem.

Dwork's definition of exponent A of a differential module P relies on the existence of a sequence of matrices $\{S_{m,A}\}_{m=1}^{\infty}$. This methodology, while effective, did not immediately reveal the relationship with decomposition, a crucial aspect for understanding the theorem's essence. This is not desirable because the mysterious computations can make it unclear to see how this theorem can be further generalized. Our aim was to establish a more intrinsic and reasonable definition of exponent and a proof of the p-adic Fuchs theorem using Frobenius antecedent and representation theory. This approach would ideally allow for a recursive and simultaneous construction of exponent and decomposition. However, it was crucial to retain the non-Liouville difference condition on the exponents, a fundamental aspect of the theory.

Christol's Example: Illuminating the Complexity

An example by Christol was particularly illustrative of the complexities involved: he began with a differential module P of rank 2 over an annulus with outer radius 1 satisfying the Robba condition, and constructed a sequence of numbers $\{r_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} r_n = 1$, such that on every open subinterval (r_n, r_{n+1}), P is an extension of rank 1 submodules. However, these decompositions do not glue up to a global decomposition of P, and exponent of P over (r_n, r_{n+1}) and $(r_{n'}, r_{n'+1})$ are "extremely different" if $n \neq n'$. This example highlighted the nuanced nature of exponents with Louville differences.

Application in de Rham Cohomology

An intriguing application of the p-adic Fuchs theorem lies in its connection with the finite dimensionality of de Rham cohomology. It is well-known that the differential equation of rank 1 over an annulus: $t\frac{dy}{dt} = ay$ has finite dimensional de Rham cohomology if and only if a is not a p-adic Liouville number. Now because it is possible to track finite dimensionality by index, which behave well under extensions of differential modules, we instantly obtain from p-adic Fuchs theorem that, for a differential module P over an annulus satisfying the Robba condition, with an exponent both non-Liouville and having non-Liouville differences, P has finite dimensional de Rham cohomology. However, to obtain a result on finite dimensionality, it is too much to ask for a direct sum decomposition globally. Interestingly, even without a global direct sum decomposition, local extensions of rank 1 subquotients can be significantly informative. Hopefully, this might be true since Christol and Mebkhout claimed a relevant result in a paper of theirs about 30 years ago. The potential for generalizing these findings to quasi-smooth analytic curves is particularly exciting, thanks in part to the semi-stable reduction theorem. This avenue of research opens new possibilities for understanding the intricate relationship between p-adic differential equations and broader mathematical concepts.

Conclusion: Reflecting on the Journey and Looking Ahead

As my time at the Institute Fourier draws to a close, I find myself reflecting on the profound journey I have undertaken in the realm of p-adic differential equations. The opportunity to study under Professor Andrea Pulita has been an invaluable experience, enriching my understanding and fueling my passion for mathematical research. This is also a journey of discovery, collaboration, and intellectual growth. It has opened new avenues for research and set the stage for future explorations in this fascinating field. I am grateful for the guidance of Professor Pulita and I look forward to continuing this journey, building upon the work we have started here.