The algebra with beautiful symmetry “Octonion”

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The concept of numbers is closely related to human life, and natural numbers 1, 2, 3, ... are especially familiar to us. Historically, human beings have expanded the notion of number sometimes for necessity and sometimes for pure interest. The integers, fractions, and real numbers were born in such a way, and they are widely known. By using them, we can quantitively measure various things (For example, number of items, population, length, wight, time, money, frequency,…). It can help people to make their life more systematic, to develop their environment and culture or to think about abstract concept. Numbers have greatly expanded the perspectives and things that humankind can do.

Numbers stated above are known from ancient times. For a long time in the history, people used only them. However, on the early modern period, new number systems emerged. This is called complex number. The complex numbers are numbers which have real part and imaginary part. For example, a complex number is written like 3+2i and in this case, real part is 3 and imaginary part is 2i. Here, i is called imaginary unit. This satisfies the property $i^2=-1$. This property cannot be satisfied by any real number. Many great mathematicians studied about complex number. They revealed its attractive properties. By their efforts, nowadays, a lot of science students more or less have dealt with complex numbers.

In 1843, mathematician Hamilton discovered the new number system which is called Quaternion. This is a generalization of complex number. Like complex number, Quartanion have imaginary part. However, they have 3 imaginary units i, j, k. Quartanions are written like 2-4i+3j+5k. In this case, real part is 2 and imaginary part is -4i+3j+5k.

Soon after the discovery of Quaternion, Octonion was founded. This is generalization of Complex number and Quaternion. Octonions have 7 imaginary units! Octonions are written like 2+i-3j+k-5p-q+2r-s.

Historically, in spite of its significancy, the Octonions have been overlooked for a long time. I think one of the reasons of it is its strange properties called Noncommutativity and Nonassociativity. Noncommutativity means that we cannot change the order of multiplication and Nonassociativity means that (ab)c and a(bc) are not equal. These properties make a calculation complicated and sometimes induce unexpected phenomenon. When confronting such a situation, some people give up using them. The mathematician J.C. Baez who writes a paper about Octonion which is full of love for it compare it to “the crazy old uncle nobody lets out of the attic”. While he compares the real number to “the dependable breadwinner”. [1]

Although the Octonions have such a complicated structure, it is a very important and very beautiful mathematical object. I will now explain two reasons for it. First, it is the biggest “normed division algebra”. “Division algebra” means “the numbers which can define division”. It is important when solving equations like “ax=b”. This is a fundamental when we study the number. “Normed algebra” means the number which we can define absolute value. By using absolute value, we can study the relationship between the number and geometry. It is known that this relationship is very fruitful for both. They greatly interact with each other. In addition to that, in fact, these two properties go well together. Thus, “normed division algebras” are marvelous mathematical objects. However, normed division algebras do not exist so many. It is not easy for numbers to simultaneously acquire the properties “Normed” and “Division”. So, if there exists normed division algebra, it is miracle matter, and we can expect there, delicate algebra and exquisite geometry. In fact, it is known that real number, complex number, Quartanion, Octonion are only normed division algebras. The Octonion includes other three normed division algebras, so it is biggest one of them.

Second, the Octonion explains well about a miraculously balanced symmetry. In late 19th century and early 20th century, Mathematicians W. Killing and E. Cartan classified "minimum unit of smooth symmetries" These symmetries are called simple lie algebras. Simple Lie algebras describe a fundamental symmetry of phenomenon around us. Rotation of sphere, classical mechanics or relative generality are examples of phenomena which can explained well by simple Lie algebras. Similarly, most of simple Lie algebras can be explained from well-known phenomena which can be constructed systematically. They are called classical ones.

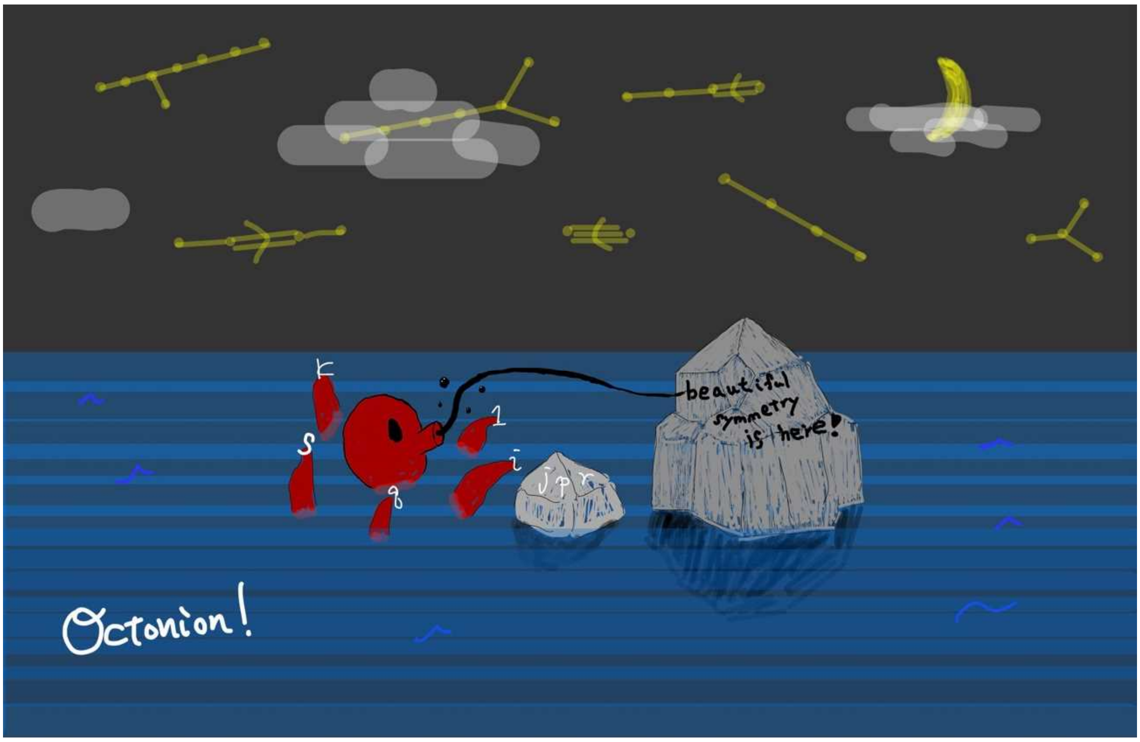
In fact, the number of simple Lie algebras which are not classical are only 5, although there exists infinitely many classical simple Lie algebra. They are called exceptional ones. At the time of discovery, the identity of exceptional ones was unknown. However, they gave a strong premonition of the existence of interesting mathematics, since their structures are very elaborated and miraculously balanced.

After that, in late 20^{th} century, eventually, it was found that the exceptional simple Lie algebras can be explained from Octonionic phenomena!

This fact means that if we can find Octonionic phenomena around us, we may see beautiful symmetry of exceptional Lie algebra and vice versa. Nowadays, in fact, exceptional Lie algebras are expected to be used in theory of everything in physics. It is proposed that supersymmetric phenomena relate deeply with Octonions. Octonions and exceptional simple Lie algebra are starting to explain our world.

In this way, the Octonion is a very attractive mathematical object. Although its rule of calculation is somewhat complicated, it is not too difficult to understand. Many people can start using it. I think Octonion is a great chance for many people to touch a highly sophisticated geometry and symmetry without the expert knowledge of mathematics. We can gain from it a new strong intuition about the natural sciences and the world around us.

Nowadays there are not yet so many Octonion users. I think, if many people become native users of Octonions, the humankinds can look further. The humankinds can find more beautiful and fundamental phenomena. Some people may come up with applications which mathematicians never thought. I think this may have potential to change our society and I think such a future is very exciting.



Reference

[1] J.C. Baez, “The Octonions”, Bulletin of the American mathematical society (2002)