Towards a Nonperturbative Many-Body Nuclear Theory:

Benchmarking FRG-DFT with the One-Dimensional Fermi Gas

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A deep microscopic understanding of quantum many-body theories would revolutionize many fields – with applications to nucleosynthesis, reliable predictions for advanced reactor designs, neutron star properties relevant for next-generation gravitational wave observations, ultracold atom experiments, and much more. Density Functional Theory (DFT) is the only approach applicable across nearly the entire nuclear chart, making the derivation of a nuclear DFT from first principles a hot topic in recent years. Modern theoretical tools such as the Functional Renormalization Group (FRG) flow equations from Quantum Field Theory [1] have enabled a first-principles formulation of DFT, known as the FRG-DFT [2]. Though the FRG-DFT has been applied to electrons with the Coulomb interaction in two and three dimensions [3, 4], a standardized theoretical benchmark comparing this method to others and a direct connection to experiments is still missing.

As apart of the 2025 University of Tokyo Research Internship Program (UTRIP), I led a collaboration between UCLA, The University of Tokyo (Japan), and Chongqing University (China) that began to address this gap by applying the FRG-DFT to the exactly solvable Gaudin-Yang model [5] – a one-dimensional Fermi gas with a strong-coupling regime, realizable experimentally with ultracold atoms [6]. By the end of the summer program, I had discovered for the first time the exact infinite hierarchy of coupled differential equations for the ground-state energy of the Gaudin–Yang model[1], and I included explicit spin-dependence to allow for straightforward extensions of this work to describe the spin-unsaturated electron gas. I have presented these results internationally, improved the skill of explaining my theories clearly in multiple ways to address language barriers, and developed a stronger work-life balance to make swift research progress in the 6-week UTRIP program while still enjoying time exploring Tokyo with my friends – all vital skills that I will use as I go on to become a professor.

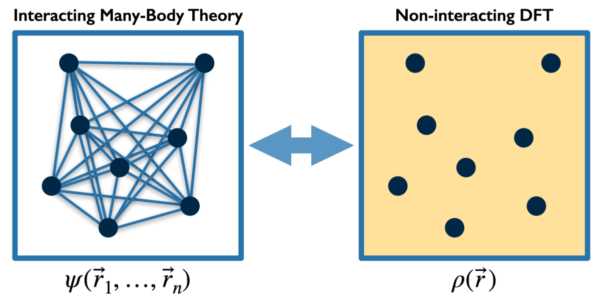

FIG. 1. The degrees of freedom of the many body wavefunction (left) scale in complexity with particle number, while the density (right) used in DFT is only ever a function of 3D space

Current methods for analyzing many-body problems in nuclear physics are often inapplicable to problems of interest: Perturbation Theory is a crowning achievement of physics in the last century, giving accurate predictions to ten decimal places for weakly-coupled theories like electromagnetism, but it performs remarkably poorly for

[1] The exact equations are not reported here as this work is ongoing with the intent to publish.

strongly-coupled theories (like the nuclear force between quarks that comprise protons and neutrons). Quantum Monte Carlo Methods and Lattice Quantum Chromodynamics calculations are fairly trustworthy, but very computationally expensive as they scale exponentially with system parameters. Density Functional Theory is a particularly appealing approach due to its success in strongly-coupled systems and computational efficiency, but suffers from not having a first principles (trustworthy) description.

The difficulty of deriving a nuclear DFT from first principles comes from precisely what makes its calculations so efficient: The Hohenberg-Kohn Theorems ensure that quantum many-body problems can be formulated solely in terms of the particle density (a function of the three dimensions of space, *x, y, z*), where the ground-state energy can be expressed as a universal functional of this density. This drastically reduces the dimensionality of problem while retaining ground state physics (FIG. 1). Though, the explicit form of the energy functional is unknown. To derive this energy as a function of particle density from first principles, the FRG-DFT makes progress by turning to powerful nonperturbative tools from Quantum Field Theory.

One important idea to simplify complex systems is to zoom out and build a theory from only the relevant low-energy physics. Many-body problems are often too complex to fully solve microscopically, so the tool of Effective Field Theory is used to integrate out the unneeded high-energy modes for the problem at hand. In this formalism, the fundamental quantity is the Effective Action, which is essentially all the allowed interactions and couplings of the coarse-grained theory – a description valid up to the maximum needed energy scale (e.g. the energy at which one makes a measurement).

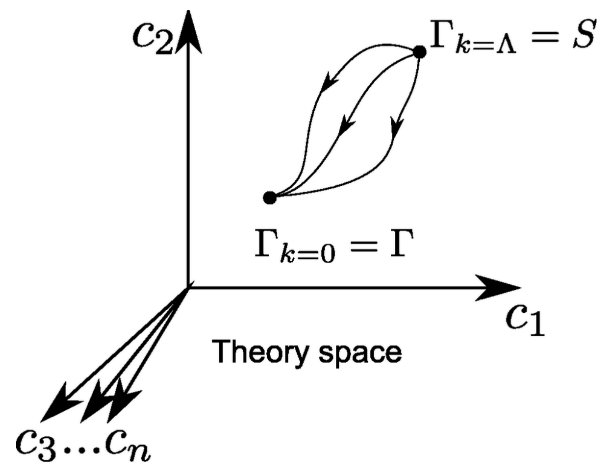

FIG. 2. Arrows indicate the RG Flow of coupling constants (axes) from a bare to effective action as high energy modes are integrated out

To succeed in writing this effective theory, one must use the Renormalization Group (RG) to zoom in and track how physics changes with energy scale. Fascinatingly, the coupling strengths of a theory – how difficult it is for particles to interact – depend on the energy scale they are measured at. This phenomenon famously causes quarks to be asymptotically free at infinite energy, but so strongly coupled at low energies they are confined. This is precisely why free quarks are not seen in everyday life, but are packaged into hadrons like protons and neutrons. The RG tracks "flows," which are how couplings evolve when integrating out the unneeded short-distance modes (FIG. 2). Naturally, Effective Field Theories are constructed from RG flows.

The language of the RG Flows is known as the Functional Renormalization Group (FRG). Functional RG flow equations are first principles and exact, but are an infinite hierarchy of coupled differential equations for the instantaneous interactions between any number of constituents (FIG. 3). Due to its

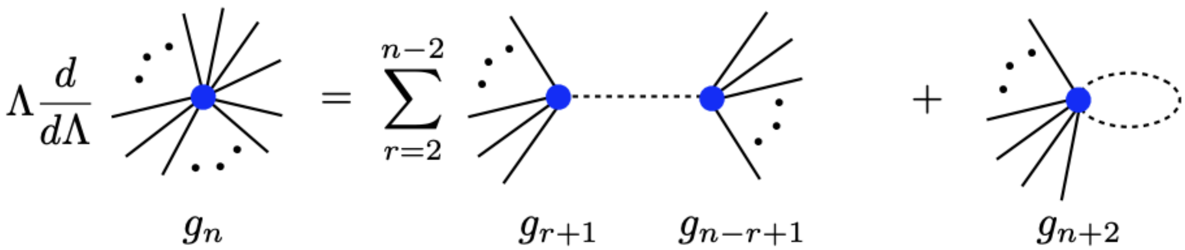

FIG. 3. Diagrammatic representation of the infinite hierarchy of FRG flow equations. The flow of the *n*-point vertex (left) depends on the flow of the *n*+2-point vertex (far right). Thus, the flow of the *n*+2-point vertex depends on the flow of the *n+4*-point vertex, and so on. In the typical 1PI scheme, legs represent particles and vertices are interactions between particles. In the 2PPI scheme, legs represent densities.

infinite nature, calculations require truncating the hierarchy, and the best way to do this is still an open problem. In the context of a first-principles DFT, the FRG explicitly computes how an Effective Action – or energy functional – evolves with scale.

In DFT, the ground state energy is a functional of the density, but its explicit form is unknown. Though, in the zero temperature limit, the Effective Action is the precisely the ground state energy. The big idea of FRG-DFT is then to write the Effective Action as functional of the density (called a 2PPI scheme, as opposed to the typical 1PI), as taking an appropriate limit then gives the ground state energy as a functional of the density. The FRG flow equations exactly describe how to arrive at the Effective Action (ground state energy) density functional from a free theory, so this method systematically includes correlation effects which are crucial at strong coupling. This workflow is shown in FIG. 4.

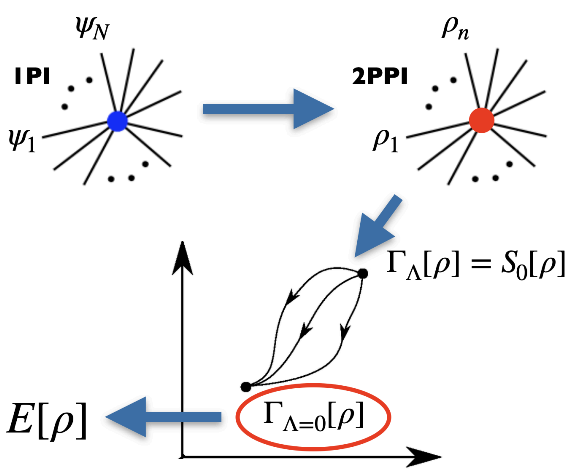

FIG. 4. Diagrammatic workflow of the FRG-DFT: write down a Quantum Field Theory in terms of densities (red vertex) instead of fields (blue vertex), use RG to flow to the effective action, then take the zero-temp limit to get the ground state energy in terms of density

To learn these useful tools and use them in practice, I drew on my knowledge from graduate coursework in Quantum Field Theory, Particle Physics, and Quantum Chromodynamics, then supplemented with specific books relevant to this work [1, 7]. Through this project, I obtained the skill of quickly mastering graduate textbooks, theses, and papers to address a new problem by learning the formalism of the FRG well enough to relax the assumptions of previous work.

Existing studies [8] have found strong evidence that the FRG-DFT is an efficient first principles way to identify the equation of state (FIG. 5). Previous work has even explored simple setups with a spin polarized gas and electrons in higher dimensions [4, 9]. Though, to develop a standardized theoretical benchmark for this novel method, one must compare to a model with an exact solution. Also keeping in mind the importance of making a direct connection to experiment, I began

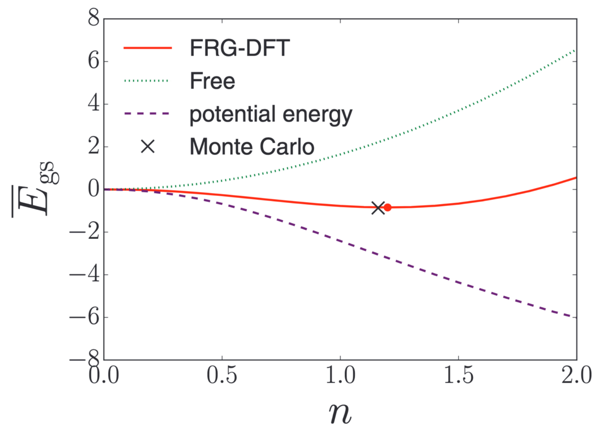

FIG. 5. FRG-DFT proof of concept from [8]: Ground State energy vs particle density for a weakly-interacting 1D system, comparing FRG-DFT predicted ground state density (red dot) to Monte Carlo (cross).

to apply the FRG-DFT to the Gaudin-Yang model [5], a one-dimensional electron gas with strong-coupling that is experimentally realizable in ultracold atom experiments [6]. Previous work [10, 11] made progress in applying this method to similar problems, but did not benchmark on an exact solution nor explain how this theory could be practically compared to experiment.

By the end of the UTRIP program, I had identified the exact infinite hierarchy of coupled FRG-DFT flow equations for the ground state energy of the Gaudin-Yang model. Developing a standardized benchmark, it is important to compare to other methods for solving the same problem. A recent work [12] studies the Gaudin-Yang model with competing approaches to the FRG-DFT. Drawing on my experience in plasma physics, I used my previous skills in accelerating physics research with Machine Learning to quickly recreate a plot from [12] which compares classical approaches like perturbation theory [13] to the exact solution for the Gaudin-Yang Model (FIG. 6). As expected, perturbation theory does poorly in the strong-coupling region.

By studying this experimentally realizable model, I have developed a direct connection to experiment for the FRG-DFT. This collaboration is currently ongoing, and I am in the process of truncating and solving the infinite hierarchy of spin-dependent flow equations for the Gaudin-Yang model. This first-principles calculation of the equation of state will be compared to perturbation theory [13] and competing approaches [12] by benchmarking on the exact solution (FIG. 6). There are many important directions of future work that must relax current assumptions. Notably, the formalism must be extended to include finite temperature effects, spin polarization [9], higher dimensional systems [4], realistic interactions [3], spontaneous symmetry breaking (external magnetic field), and more. Though, my contributions have established a clear theoretical benchmark for FRG-DFT and lay the foundation for its extension to realistic higher-dimensional many-body systems – crucial next steps towards a first principles nonperturbative many-body nuclear theory.

An important aspect of this program was being able to present technical

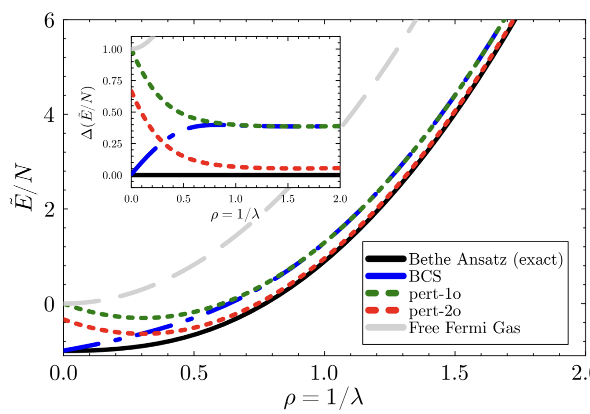

FIG. 6. Ground State energy vs density for the Gaudin-Yang Model with exact solution. Low density corresponds to strongly coupled, and perturbation theory poorly describes the energy in this strong-coupling region

scientific work effectively. My former experience in presenting my research at the most prestigious international conference in plasma physics (the APS-DPP) and lecturing at Perimeter's PSI START Satellite program in Canada well prepared me to lead meetings for discussion in this international collaboration. While preparing for these meetings, I thought critically about how to convey my theories clearly as English is not typically the first language, and I developed the valuable skill of clearly explaining my research in multiple ways across language barriers.

In addition, this collaboration has broadened my perspective on the international nature of science and led to deeper connections with colleagues than I had thought possible. My brief time researching in Japan inspired me to seek longer-term opportunities in physics abroad where I can immerse myself in a new culture and build more lasting connections with like-minded peers.

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