A theoretical model was proposed in 2013 by Moriya et al. predicting the light curves of interaction powered supernovae, potentially applicable to type IIn events. A possible improvement to that model for the early phase is presented. This improvement seems to be supported by recalculated values of parameters derived using the original model, that when compared, appear to be slightly more reasonable. More importantly, it could probably be used for more accurate modeling of such events.

I. Introduction

Massive stars that close their life circles as supernovae take many different evolutionary paths. As a result of that, there are several types of supernovae depending on circumstances. Thus, the study of each type gives insight to the unique conditions that lead to its particular characteristics.

The goal of this work is the description of a certain type of supernovae, those classified as IIn (Filippenko 1997). Their main characteristic is the existence of narrow hydrogen emission lines in their spectra, caused by dense circumstellar media surrounding the star. The existence of circumstellar media is proposed to be related with severe mass loss by the stars for years before the final explosion and their luminosities are suggested to emerge from the interaction of supernova ejecta with the circumstellar medium. The exact mechanism resulting in the formation of the circumstellar medium and the progenitors of these supernovae are still a matter of debate. Therefore, proper modeling of such explosions, resulting in reliable values about their characteristic quantities is essential in order to deepen our understanding.

A theoretical model was proposed in 2013 (Moriya et al. 2013) describing the interaction of supernova ejecta with a dense ambient medium, allowing one to construct the expected bolometric light curve of such an event. That model could satisfactorily reproduce some of the properties of type IIn supernovae's light curves, mainly their slowly fading luminosities, and could be fitted to observational data, resulting in relatively reasonable values for the fitted parameters, compared to values derived through different methods, or typically expected values. One thing seemed kind of odd though, that being the calculated initial ejecta kinetic energies. The energies calculated by that model were of the order of $10^{52} \text{erg}$ . Given the fact that these supernovae are considered core collapse events, in which ejecta usually reach
energies of the order of $10^{54}$ erg, the derived energies appear to be too high. So, more specifically, an aim of this work is the improvement of that model towards this direction.

This report is structured as follows: first of all, the model by Moriya et al. is presented (their early solution in particular). After that, some properties of the model are highlighted and an investigation of the energy distribution in that model is presented, which seems to lead to a result somewhat improving the model. Later on, the results are discussed and finally, this report is concluded.

II. Model by Moriya et al.

During a supernova explosion, dense ejecta are accelerated to highly supersonic speeds. These ejecta collide with the less dense material surrounding the star, resulting in the formation of a shock wave in it. Of course, ejecta are pushed back as well and another shock wave is also formed there (Figure 1). Chevalier found a self-similar solution to describe such shock waves when there are not any energy losses (Chevalier 1982). In the case of IIn though, the adiabatic expansion approximation is not good, because energy losses due to radiation should be important.

Instead of that, Moriya et al. assumed that radiative cooling is efficient enough to cool down the shocked gas and prevent it from expanding too much. Because of that, the interaction region forms a significantly thin shell as compared to its radius. Based on that and by additionally assuming that the flow is spherically symmetric and that the velocity of the thin shell is much higher than the stellar wind velocity (i.e. the circumstellar medium can be considered ambient) they devised a model describing the time evolution of the explosion and finally predicting the bolometric light curve it would give.

Regarding the density structures of the two colliding regions, a double power-law density profile was assumed for the ejecta, with $\rho_{ej} \propto r^{-\delta}$ for inner ejecta and $\rho_{ej} \propto r^{-n}$ for outer ($r$ being the distance from the center of the star), based on numerical simulations (Matzner & McKee 1999). The inner density slope $\delta$ is $\sim 0 - 1$, while the outer density slope $n$ depends on the progenitor star and it is much steeper. Typical values are $\sim 10$ for supernova Ib, Ic and Ia progenitors, while a value of $\sim 12$ is expected for red supergiants. Considering the ejecta expansion homologous (i.e. each fluid parcel is moving at a constant velocity and that velocity is proportional to the radius), this density structure can be expressed as (see appendix):

$$\rho_{ej}(r, t) = \begin{cases} \frac{1}{4\pi(n-\delta)} \frac{(3-\delta)(n-3)M_{ej}}{(3-\delta)(n-3)M_{ej}}^\frac{n+3}{2} \frac{1}{(3-\delta)(n-3)M_{ej}}^\frac{n+3}{2} r^{-3n-\delta} & (r > u_t t) \\ \frac{1}{4\pi(n-\delta)} \frac{(3-\delta)(n-3)E_{ej}}{(3-\delta)(n-3)M_{ej}}^\frac{\delta+3}{2} \frac{1}{(3-\delta)(n-3)M_{ej}}^\frac{\delta+3}{2} r^{-3n-\delta} & (r < u_t t) \end{cases}$$

where $M_{ej}$ and $E_{ej}$ stand for the initial total mass and kinetic energy of the ejecta and $u_t$ is the constant velocity of the “knee” of the structure:

$$u_t = \frac{(3-\delta)(n-3)E_{ej}}{(3-\delta)(n-3)M_{ej}}^\frac{1}{2}$$

Moreover, a single power-law density structure was assumed for the circumstellar medium, $\rho_{CSM} = Dr^{-\gamma}$, with a constant $D$.

Moving on, one can write the equation for conservation of momentum as:
with \( u_{sh} \) being the shell’s velocity, \( u_{ej} \) being the ejecta velocity right behind the shell and \( M_{sh} \) being the mass of the shell.

\[
M_{sh} \frac{du_{sh}}{dt} = 4\pi r_{sh}^2 \left[ \rho_{ej} \left( u_{ej} \right)_{r=r_{sh}} - u_{sh} \right]^2 - \rho_{CSM} u_{sh}^2 
\]

(3)

\( M_{sh} \) as a function of time and \( r_{sh} \) can be found for the epoch that the outer ejecta are interacting with the circumstellar medium:

\[
M_{sh} = \frac{4\pi D}{3-s} r_{sh}^{3-s} + \frac{1}{(n-\delta)(n-\delta-3)M_{ej}} \left( \frac{t}{n-\delta} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-3} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-1} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-3} \right)^{n-\delta} 
\]

(4)

An exact solution of equation (3) can be found for that epoch:

\[
r_{sh} = \left[ \frac{(3-s)(4-s)}{4\pi D(n-4)(n-3)(n-\delta)} \right]^{1/4} \left[ \frac{2(5-\delta)(n-5)E_{ej}}{(n-\delta)(n-3)M_{ej}} \right]^{1/4} \left( \frac{t}{n-\delta} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-3} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-1} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-3} \right)^{n-\delta} 
\]

(5)

This solution is valid only for \( s < 3 \) (> 3 would mean the shell is accelerating).

Given the time evolution of the shell, the expected radiated power can be approximated assuming that the luminosity of the supernova is fueled solely by the ejecta kinetic energy. Moriya et al. stated that the available power follows the next formula:

\[
d\dot{E}_{ej} = 4\pi r_{sh}^2 \rho_{CSM} u_{sh}^2 
\]

(6)

A portion of this power is radiated. Derivation of the exact efficiency of the conversion requires knowledge of the radiative processes in effect in the shell and it is not trivial. In general, it could depend on many parameters, including elemental abundances and, because of that, vary in time. Moriya et al. assumed a constant energy conversion efficiency coefficient with an arbitrary, yet reasonable value (\( \epsilon = 0.1 \)) and formulated the bolometric luminosity as:

\[
L = \epsilon \frac{d\dot{E}_{ej}}{dt} 
\]

(7)

Using formula (4), one gets that the early bolometric light curve such an event would give can be expressed as a single power-law function:

\[
L = L_1 t^a 
\]

(8)

where:

\[
a = \frac{6s-15+2n-ns}{(n-s)} 
\]

(9)

\[
L_1 = \frac{\epsilon}{4\pi D} \frac{n^s}{(n-3)^3} \left( \frac{n-\delta}{n-\delta-3} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-1} \right)^{n-\delta} \left( \frac{n-\delta}{n-\delta-3} \right)^{n-\delta} 
\]

(10)

The above formulas can be used to fit observed early light curves (Moriya et al. 2014). Then, by assuming \( n \) and \( \delta \), the density slope of the circumstellar medium can be constrained. With additional information regarding the velocity of the shell, \( D \) can be constrained as well (actually, the shell’s velocity as a function of time needs to be fitted, but since the time dependency of that velocity is small, a single value is enough). Knowledge of the circumstellar medium density structure is important in order to understand what are the phenomena taking place in the star before the explosion and to constrain the type of the progenitor. Moreover, in principle, the value used for \( n \) can also give us information regarding the progenitor type.

If the circumstellar medium’s density structure is known, assuming the wind velocity \( u_w \) before the explosion one can approximate the mass loss rate before the explosion:

\[
\dot{M}(t') = 4\pi D u_w^{3-s} t'^{2-s} 
\]

(11)

where \( t' \) is the time before the explosion.

At this point, ejecta kinetic energy and mass are degenerate, meaning that one has to be assumed to get the other. That does not affect any results regarding the circumstellar medium density structure. It is necessary to break that degeneracy though, in order to estimate when the interaction region has reached down to the inner ejecta, since after that the above solution is no longer valid. Moriya et al. assumed \( \dot{M}_{ej} \) for that reason and calculated the ejecta energies. The energies derived that way are those that seemed to be too large.
The same procedure can be followed to describe the interaction of the inner ejecta with the circumstellar medium. In that case, an exact solution of the momentum equation was not available, but an asymptotic one was derived, for when most of the ejected mass is inside the shell. Investigation of this epoch could break the degeneracy described before. A problem with that is that it would require sufficient data about the luminosity of the supernova for both epochs, which is not usually the case.

III. Self-similarity of the early solution

The solution describing the time evolution of the thin shell during the interaction of the outer ejecta with the circumstellar medium shows some kind of self-similarity. In fact, dimensional arguments can be used to derive this solution (see appendix). That self-similarity can be helpful, not that much in solving the momentum equation (since the solution derived by Moriya et al. is not that hard to get) but because of some properties incidental to it, as shown below.

The main consequence of the self-similar nature of the solution that will be used is that the ratio of the ejecta density right before the shell to the density of the circumstellar medium right after the shell is constant in time:
\[
\rho_{ej} \frac{\rho_{CSM}}{\rho_{ej}} = \text{const.} \quad (12)
\]
One can easily see the validity of this statement by directly calculating the density ratio for \( r = r_{sh} \), using formula (5) for \( r_{sh} \).

Moreover, due to the radius of the shell being described by a power-law function of time, the velocity is also a power-law function of time:
\[
u_{sh} = \frac{dr_{sh}}{dt} = \frac{n-3}{n-3} r_{sh}
\]
\[
u_{sh} = \frac{n-3}{n-3} u_{sh} \quad (13)
\]
which means that the ratio of the ejecta velocity just behind the shell over the velocity of the shell (and the material inside it) is time invariant as well.

These two properties give us a useful result: any two products of linear combinations of \( u_{sh}, u_{ej}, \rho_{ej}, \rho_{CSM}, r_{sh} \) with power units have a time invariant ratio. The general form of such a product is:
\[
P \propto r_{sh}^2 \left( \rho_{ej} + B \rho_{CSM} \right)^3 \prod_{i=1}^{n} \left( C_i u_{ej} + G_i u_{sh} \right)
\]
and the exact value of such a ratio can be easily calculated using formulas (12) and (13).

IV. Investigation of energy distribution during the early phase of the evolution of the shell

As shown earlier, Moriya et al. approximated the bolometric light curve of an interaction powered supernova by assuming that a constant portion of the power available to the shell was emitted as electromagnetic radiation at any given time. They considered that the shell was fueled by the ejecta kinetic energy and calculated the power provided using formula (6). Although the kinetic energy carried by ejecta as they enter the interaction region is indeed the only energy that could dissipate in the narrow shocked region, the formula used does not express that power, but rather the power consumed by the circumstellar material swept inside the shell in order to get accelerated to the velocity of the shell.

In order to calculate the actual energy provided to the shell in the unit of time, one needs to integrate the macroscopic kinetic energy flux on the interface between the ejecta and the shell, with respect to the fact that the interface is moving:
\[
P_{ej-sh} = \oint_{r=r_{sh}} \rho_{ej} u_{ej}^2 \left( u_{ej} - u_{sh} \right) \cdot d\alpha
\]
\[
P_{ej-sh} = 4 \pi r_{sh}^2 \rho_{ej} u_{ej}^2 \left( u_{ej} - u_{sh} \right) \quad (14)
\]
calculated for \( r = r_{sh} \). Since the energies given by formulas (6) and (14) correspond to different processes, it is not obvious that they should be correlated in any way.
By taking in account the result shown in the previous section though, it is easy to see that these two energies are not just correlated, but proportional:

\[ P_{\text{ej-sh}} = \frac{(n-4)(n-3)^2}{(4-s)(n-3)^2} \frac{dn_{\text{sh}}}{dt} \]  

(15)

That means that changing the expression used to calculate the available energy does not change the expected light curve but by a constant factor that can be absorbed in the efficiency coefficient. Assuming a constant conversion efficiency coefficient \( \epsilon \) and using equation (14), gives a light curve compatible with equation (7) if \( \epsilon \) is:

\[ \epsilon = \frac{(n-4)(n-3)^2}{(4-s)(n-3)^2} \bar{\epsilon} \]  

(16)

and the values for all other parameters are unchanged. In figure 2 the ratio \( \bar{\epsilon}/\epsilon \) as a function of \( s \) for three characteristic values of \( n \) is shown. One can see that \( \epsilon \) is always larger, which means that the original formula tends to underestimate the available energy.

![Image](image_url)

**Figure 2** - \( \bar{\epsilon}/\epsilon \) (\( \bar{\epsilon}/\epsilon \)) as a function of \( s \) for three characteristic values of \( n \).

Since the choice of \( \epsilon \) is arbitrary though, the above result alone is not that important. Even if formula (6) underestimates the available energy, if the conversion efficiency is overestimated, zero progress was made. Some sort of constraint on \( \epsilon \) is needed, but deriving one by studying the radiative processes in effect is, as mentioned before, hard.

Instead of that, the power required by the shocked gas in the shell for its macroscopic motion can be easily calculated. Then, the difference between \( P_{\text{ej-sh}} \) and that power would give the energy left to heat up the shell and to be radiated, giving a maximum for the luminosity. By further assuming that the cooling is very effective, this difference could be taken to be equal to the luminosity. In any case, the energy radiated needs to be comparable to the available energy, because otherwise the expansion would be described better by the adiabatic model.

Given the time evolution of the shell, and using equations (12) and (13), one gets:

\[ \frac{d}{dt} \left( \frac{1}{2} M_{\text{sh}} U_{\text{sh}}^2 \right) = \frac{(n-3)}{2(n-3)^2} \left[ n - 2s + 3 - \frac{(n-3)^2}{(n-4)^2} P_{\text{ej-sh}} \right] \]  

(17)

and the remaining energy is:

\[ T = P_{\text{ej-sh}} - \frac{d}{dt} \left( \frac{1}{2} M_{\text{sh}} U_{\text{sh}}^2 \right) \]  

(18)

which means that the maximum value of \( \bar{\epsilon} \) (the conversion efficiency coefficient used for energy calculated with equation (14)) is given by the following equation:

\[ \bar{\epsilon} = \left[ 1 - \frac{(n-3)}{2(n-3)^2} \left( n - 2s + 3 - \frac{(n-3)^2}{(n-4)^2} \right) \right] \]  

(19)

In figure 3 \( \bar{\epsilon} \) is plotted as a function of \( s \) for three characteristic values of \( n \). The fact that the maximum efficiency (for energy expressed using equation (14)) is of the order of \( \sim 0.6 \) indicates that the initial model indeed underestimated the luminosity of the explosion, as such values require \( \epsilon \) (the efficiency coefficient used when energy is expressed using equation (6)) of the order of \( \sim 2.5 \) in order to give similar results.

![Image](image_url)

**Figure 3** - \( \bar{\epsilon} \) as a function of \( s \) for the same values of \( n \).
In order to check the effect of the proposed correction on the model, ejecta kinetic energies and mean mass loss rates were recalculated for two events, SNe 2005ip and 2006jd, already fitted by Moriya et al. (Moriya et al. 2013, see their references for more works regarding those events). The values are presented in tables 1 and 2.

Table 1 - SN 2005ip

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<tr>
<td>n</td>
<td>s</td>
<td>$E_{ij}$ (ergs)</td>
<td>$&lt; M &gt;$ (M$_\odot$·yr$^{-1}$)</td>
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<tr>
<td>Moriya et al. 2013</td>
<td>10</td>
<td>2.3</td>
<td>1.3·10$^{52}$</td>
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<tr>
<td></td>
<td>12</td>
<td>2.4</td>
<td>1.5·10$^{52}$</td>
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<tr>
<td>Using the correction presented here, $\epsilon = 0.1$</td>
<td>10</td>
<td>2.3</td>
<td>8.6·10$^{51}$</td>
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<tr>
<td></td>
<td>12</td>
<td>2.4</td>
<td>1.0$^{52}$</td>
</tr>
<tr>
<td>Same as above, but for maximum $\epsilon$</td>
<td>10</td>
<td>2.3</td>
<td>5.1·10$^{51}$</td>
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<td></td>
<td>12</td>
<td>2.4</td>
<td>6.8·10$^{51}$</td>
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Table 2 - SN 2006jd

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<td>n</td>
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<tr>
<td>Moriya et al. 2013</td>
<td>10</td>
<td>1.4</td>
<td>1.2·10$^{52}$</td>
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<tr>
<td></td>
<td>12</td>
<td>1.6</td>
<td>1.3·10$^{52}$</td>
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<tr>
<td>Using the correction presented here, $\epsilon = 0.1$</td>
<td>10</td>
<td>1.4</td>
<td>8.4·10$^{51}$</td>
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<td></td>
<td>12</td>
<td>1.6</td>
<td>9.3·10$^{51}$</td>
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<tr>
<td>Same as above, but for maximum $\epsilon$</td>
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<td>1.4</td>
<td>4.9·10$^{51}$</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>1.6</td>
<td>6.2·10$^{51}$</td>
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The recalculated energies seem to be corrected on the right direction, but they are still high, as the dependency of $E_\odot$ on $\epsilon$ is weak. Changing the assumed $M_\odot$ value (here and in Moriya et al. 2013 it is taken to be 10 M$_\odot$) could affect them more, but there are no physical arguments regarding how it should be changed.

The mass loss rates are more affected, since one can show that $M \propto \epsilon^{-1}$. This shows that it is indeed important to constrain $\epsilon$, since $M$ is one of the important parameters in order to acquire better understanding of the conditions that lead to a IIn event. These values are also dependent on the assumed stellar wind velocity. This parameter is actually sensitive to the type of the progenitor star. The fact that our new values are quite lower compared to those calculated by Moriya et al. could indicate that the assumed value ($u_\infty = 100$ km s$^{-1}$) is an underestimation, which could support the scenarios with more massive progenitors. On the other hand, a more massive progenitor could require larger values for the total ejecta mass, which would result in higher values for the total energy. Since the calculated energy is already a bit extreme, further optimization of the model is required to strongly support such scenarios.

V. Discussion

The correction proposed here on the initial model still leaves room for further improvement on the modeling of interaction powered supernovae. That can be seen by the fact that the new values for energy, although maybe more reasonable, still seem a bit off. Still, a non-arbitrary but based on physical arguments value for the efficiency coefficient can be derived through the analysis proposed here, reducing the number of assumed parameters.

In any manner, since the values calculated here are not completely satisfactory either, less rough approximations need to be made to increase the plausibility of the model.

One of the main factors that could affect the evolution of the shell in many ways is its thickness. The rate at which it expands is directly related to the energy stored in it as heat. The amount of energy stored as heat is in turn related to the efficiency of the radiative cooling. Different expansion rates would be the result of different efficiencies, which means that one could probably insert the information of the radiative processes in effect in the shell in $\epsilon$ through that rate.
The thickness of the shell affects the momentum equation (3) as well. Of course, if the shocked areas expand a lot, the evolution of the interaction would be probably described better by the solution for the adiabatic case. But it would be possible to have a small, yet significant width. In that case, if a single constant in space velocity is assumed for the shell and \( r_{sh} \) shows, for example, at the contact discontinuity (\( r_{sh} \) should be a characteristic radius of the shell, relatively to which the distances of its borders are expressed), the flows of momentum in the right-hand part of equation (3) would have to be calculated not on \( r_{sh} \) but on \( r_{sh} + dL_1 \) and \( r_{sh} - dL_2 \), with \( dL_1 + dL_2 \) being the width of the shell.

One last effect that a finitely wide, expanding shell would have is that it would no longer have a constant in space velocity, but more likely a space-depending profile. It is possible to use some proper average velocity, but that would affect both the momentum equation and the expression for the power used by the shell, probably decreasing it.

Lastly, a second thing one could take in account in order to make more precise calculations of the radiated energy is that there are probably other radiation sources as well, like radioactive decay. Even if the dominant power source is the interaction between ejecta and circumstellar medium, other effects are not necessarily insignificant. The model will tend to underestimate the luminosity of a given object because of that.

VI. Conclusions

The main aim of this work was to improve the model proposed by Moriya et al. which seemed to be on the right track, but when used on observational data gave somewhat extreme energy values. Although the problem was not entirely solved, the results presented here are correcting those values in the right direction. One of the assumed values of the initial model was constrained. Lastly and most importantly, the results presented here can probably be used for further optimization of the model.

Acknowledgments

I would like to thank Professor Toshikazu Shigeyama for his help and guidance on my project, Yuki Takei and my fellow lab mates for the warm hospitality, the International Liaison Office for the administrative help and care and the Graduate School of Science of the University of Tokyo for the generous scholarship. I would also like to thank Professor Nektarios Vlahakis, to whom I am indebted for his help and guidance during my preparation for the program.

Appendix

In this appendix, some technical details in deriving the early solution described by Moriya et al. are presented. Not all of them seemed trivial to me. That is why this section was written.

Ejecta density structure

The assumptions made regarding the ejecta were homologous expansion and a double power-law density profile. As a result of these two conditions, density can be expressed as a function of position and time by equations (1), (2).

In more detail, homologous expansion dictates that each fluid parcel moves at a constant velocity, radially and the velocity field is proportional to \( r \):

\[
u_{ej} = \frac{r}{t}
\]  
(A1)

In this case, each fluid parcel is identified by its velocity (since it is constant) so, for some procedures, it is convenient to use velocity as a coordinate instead of position (Langrangian specification of the flow field).

A double power-law density profile means:
\[
\rho_q(r, t) = \begin{cases} 
C_{\text{out}}(t)r^{-n} & (r > r_1(t)) \\
C_{\text{in}}(t)r^{-\delta} & (r < r_1(t)) 
\end{cases} \\
\rho_q(r, t) = \begin{cases} 
C_{\text{out}}(t)r^{-n} & (r > u_1t) \\
C_{\text{in}}(t)r^{-\delta} & (r < u_1t) 
\end{cases}
\] (A2)

with \(r_1\) and \(u_1\) being the position and velocity of the “knee” of the structure, the point the two slopes meet, taking in account (A1).

In order to find \(C_{\text{out}}, C_{\text{in}}\) and \(u_1\), three equations are required. Firstly, density needs to be described by a continuous function, so on \(r = r_1\):

\[
\rho_q|_{r=r_1} = \rho_q|_{r=r_1} 
\]

\(C_{\text{out}}(t)(u_1t)^{-n} = C_{\text{in}}(t)(u_1t)^{-\delta} \Rightarrow (u_1t)^{-n-\delta} = \frac{C_{\text{out}}(t)}{C_{\text{in}}(t)} \Rightarrow u_1 = \frac{1}{7} \left(\frac{C_{\text{out}}(t)}{C_{\text{in}}(t)}\right)^{\frac{1}{n-\delta}} \) (A3)

Moreover, the total ejecta mass equals to the integral of density from \(r = 0\) to \(r = r_{\text{max}}\), the radius of the outermost ejecta layer. Since the outer ejecta density structure is steep, \(r_{\text{max}} \rightarrow +\infty\) is a good approximation and one gets:

\[
M_{\text{ej}} = \int_0^{r_{\text{max}}} dv \rho_q = 4\pi \left[ \int_0^{u_1 t} dr C_{\text{in}}(t)r^{2-\delta} + \int_{u_1 t}^{+\infty} dr C_{\text{out}}(t)r^{2-n} \right] \Rightarrow \frac{M_{\text{ej}}}{4\pi} = C_{\text{in}}(t) \frac{t^{2-\delta}u_1^{3-\delta}}{3-\delta} + C_{\text{out}}(t) \frac{t^{2-n}u_1^{3-n}}{n-3} \) (A4)

Similarly, the ejecta total kinetic energy is:

\[
E_{\text{ej}} = \int_0^{+\infty} dv \frac{1}{2} \rho_q u^2 = 2\pi \left[ \int_0^{u_1 t} dr C_{\text{in}}(t) \frac{r^{4-\delta}}{3-\delta} + \int_{u_1 t}^{+\infty} dr C_{\text{out}}(t) \frac{r^{4-n}}{3-n} \right] \Rightarrow \frac{E_{\text{ej}}}{2\pi} = C_{\text{in}}(t) \frac{t^{4-\delta}u_1^{5-\delta}}{5-\delta} + C_{\text{out}}(t) \frac{t^{4-n}u_1^{5-n}}{5-n} \) (A5)

where equation (A1) was used. Solving the resulting \(3 \times 3\) equation system gives equations (1) and (2).

The above show that equation (1), (2) describe the free homologous expansion of a gas. As the material ejected by the star collides with the ambient medium, the information of the collision is carried by the shock waves. Therefore, an area not yet reached by the shockwaves moves unaffected by the collision and that is why those expressions can be used.

**Momentum equation**

It was mentioned above that conservation of momentum in the shell’s neighborhood is expressed by equation (3). That equation states that the shell gains momentum through the deceleration of ejecta. That momentum is used to accelerate the circumstellar material swept inside the shell. If there is not enough momentum (or too much), the shell is decelerated (or accelerated) in order to cover the difference. Although legit, this expression is not the usual form of the conservation law. In fact, if the fundamental conservation law was integrated around the shell, it would give:

\[
\frac{dM_{\text{sh}}}{dt} + M_{\text{sh}} \frac{du_{\text{sh}}}{dt} = 4\pi r_{\text{sh}}^2 \left( \rho_{\text{ej}} \left( u_{\text{sh}} \right) |_{r=r_{\text{sh}}} - u_{\text{sh}} \right) + \rho_{\text{CSM}}(u_{\text{sh}} - u_w) \]

(A6)

If the fundamental mass conservation equation is integrated, it gives:

\[
\frac{dM_{\text{sh}}}{dt} = 4\pi r_{\text{sh}}^2 \left( \rho_{\text{ej}} \left( u_{\text{sh}} \right) |_{r=r_{\text{sh}}} - u_{\text{sh}} \right) + \rho_{\text{CSM}}(u_{\text{sh}} - u_w) \]

(A7)

Using equation (A7) in (A6) one takes:

\[
\frac{dM_{\text{sh}}}{dt} = 4\pi r_{\text{sh}}^2 \left( \rho_{\text{ej}} \left( u_{\text{sh}} \right) |_{r=r_{\text{sh}}} - u_{\text{sh}} \right)^2 - \rho_{\text{CSM}}(u_{\text{sh}} - u_w)^2 \]

(A8)

which becomes equation (3) if \(u_{\text{sh}} \gg u_w\).

**Shell’s mass**

In order to solve equation (3), it is needed to express the shell’s mass as a function of its radius and time. There are at least two ways to do that.

The first way, the one used by Moriya et al., is to integrate the parts of the density structures that are in the shell. Since mass is conserved, by integrating the ejecta density structure from the shell to the maximum radius it would expand if there was no material around the star one gets the ejecta mass inside the shell. In a similar manner, if the circumstellar medium’s density structure is integrated from the progenitor’s radius to the shell, the mass swept inside the shell is found. According to these, we get:

\[
M_{\text{sh}} = \frac{\int_{r_{\text{sh}}}^{r_{\text{max}}} r \rho_{\text{CSM}} du + \int_{r_{\text{sh}}}^{r_{\text{sh}}} \rho_{\text{ej}} du}{r_{\text{sh}}} \]

(A9)

These two integrals can be calculated and so the shell’s mass can be found. If \(r_{\text{sh}} \gg r_p\)
and \( r_{\max} \gg r_{\text{sh}} \) (both true, since the radius of the shell increases very fast during the beginning of the explosion and the outer ejecta layers expand at even higher velocities) \((A9)\) gives \((4)\).

Another equivalent way to find the shell’s mass, is by integrating equation \((A7)\) in time from \( t = 0 \) to \( t \), taking \( M_{\text{sh}}(t = 0) = 0 \):

\[
M_{\text{sh}} = 4\pi \int_0^t dt' r_{\text{sh}}^2 \rho_{eJ} \left( u_{ej} \right)_{r = r_{\text{sh}}} - u_{ej} \right) + \\
\rho_{\text{CSM}}(u_{\text{sh}} - u_w) \quad \text{(A11)}
\]

As expected, if \( u_{\text{sh}} \gg u_w \), \((A11)\) gives \((4)\).

**Solution of momentum equation**

Equation \((3)\), using \((4)\), becomes a non-linear equation of \( r_{\text{sh}} \) and it is not easy to derive its general solution. The fact that there are no constant terms suggests that the equation might have power law solutions. By setting \( r_{\text{sh}} = At^k \), one gets expression \((5)\) for \( r_{\text{sh}} \) by demanding \( A \) to be a constant.

Instead of that, one can use the fact that the only parameters with dimensions on which the solution should depend are \( D \) and \( \frac{1}{4\pi(n-\delta)} \left[ \frac{2(5-\delta)(n-5)E_0}{(3-\delta)(n-3)M_0} \right]^{\frac{n-3}{n-2}} = D_0 \). We have that:

\[
[D] = [m(\text{mass})][l(\text{length})]^{s-3} \quad \text{(A12)}
\]
\[
[D_0] = [m][l]^{n-3}[t(\text{time})]^{3-n} \quad \text{(A13)}
\]

Characteristic lengths and timescales cannot be created using these parameters, which indicates self-similarity. The only combination of them with length units is \( \left( \frac{D_0}{D} \right)^{\frac{1}{n-2}} t^{\frac{n-3}{n-2}} \) and so, we expect:

\[
r_{\text{sh}} \propto \left( \frac{D_0}{D} \right)^{\frac{1}{n-2}} t^{\frac{n-3}{n-2}} \quad \text{(A14)}
\]

That way, we can find the time dependence of the solution without the equation. Using \((A14)\) in \((3)\), we get \((5)\). The same dimensional argument stands in the solution by Chevalier (Chevalier 1982) for the adiabatic case on the contact discontinuity and that is why the time dependencies in those two cases are the same.

**References**