Genralised Floquet Time Crystal Systems

Gautam Kamalakar Naik
Physics Department, Indian Institute of Technology (Banaras Hindu University), Varanasi - 221005, India
Katsura Laboratory, Department of Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
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Time translational symmetry breaking is a possibility explored only in the recent past. In this report, we first review a paper by Else, Bauer and Nayak which defines the notion of time translational symmetry breaking in quantum systems and gives the analytical and numerical study of a spin ½ model that shows this behavior. Secondly we extend this idea to analytically and numerically show that it is possible to have time translational symmetry breaking in higher spin systems.

I. INTRODUCTION

A. Spontaneous Symmetry Breaking (SSB)

Spontaneous symmetry breaking is a process where a symmetric state of a system spontaneously changes to an asymmetric state. This occurs when the underlying laws of a system obeys a certain symmetry but the ground states or the low energy states of the system do not obey the same symmetry. The simplest example for this phenomenon is that of crystals. A volume of gas in a uniform potential has continuous translational symmetry. If this system cools down and suppose the gas now crystallizes, then, the system no more has continuous translational symmetry but now has discrete translational symmetry. Continuous translational symmetry is said to be broken in this case. Another example is that of the Ising model. The Ising Hamiltonian (for a chain of spin ½ particles) $H_{\text{Ising}} = J \sum_i S_i S_{i+1}$ is symmetric under the simultaneous flip of all the spins of the system. In the ferromagnetic case ($J < 0$), there are low energy states with non-zero magnetization. Under the action of the symmetry (i.e. under spin-flip), we obtain other low energy states which have opposite magnetization. For finite chains, symmetric and anti-symmetric superpositions of such states are the eigenstates. These superpositions, also known as cat states, cannot be distinguished by any local measurements and are extremely sensitive to external fields. Even infinitesimal fields cause the system to decohere and give a state with magnetization along the external field. SSB in this system may be defined in terms of the low energy eigenstates being cat states or to the development of magnetization on application of an infinitesimal external field.

There are many other examples of SSB in particle physics and condensed matter physics - gauge symmetry breaking, electromagnetic gauge symmetry breaking, chiral symmetry breaking and Euclidean symmetry breaking to name a few. Many examples we have for SSB seems to suggest that for every symmetry we can think of, we can come up with a system which breaks that symmetry. In this report we are interested in time translational symmetry breaking.

B. Time Translational Symmetry Breaking (TTSB)

The concept of TTSB was first described by Franck Wilczek in 2012 [2] [3]. Systems which break time translational symmetry have been called ‘Time Crystals’ in analogy with ordinary crystals which break continuous translational symmetry.

The definition of TTSB is not straightforward. As discussed by Else, Bauer and Nayak (EBN) in [1], the most obvious definition and its modifications, all based on the expectation values and correlation
functions of the thermal equilibrium state $\rho = e^{-\beta H}$ must be ruled out due to no-go theorems [4]. These theorems show that it is impossible to have TTSB in equilibrium states and hence suggest the need to explore the possibility of having TTSB in non-equilibrium states. Even in the case of symmetries other than that of time translation, when we consider spontaneously broken symmetries, we look at nonequilibrium systems which show ergodicity breaking and have life times which diverge with the system size. It should be possible to have systems which show analogous behavior for TTSB.

Taking these into consideration the EBN paper gives two equivalent definitions of TTSB. Considering a Hamiltonian $H(t)$ periodic with a period $T$ with its corresponding unitary as $U(t, t_i)$ for time evolution from time $t_i$ to time $t_f$, TTSB is defined as:

**TTSB-1:** TTSB occurs if for each $t_i$, and for every state $|\psi(t_i)\rangle$ with short-ranged correlations, there exists an operator $\Phi$ such that $\langle \psi(t_i + T) | \Phi | \psi(t_i + T) \rangle \neq \langle \psi(t_i) | \Phi | \psi(t_i) \rangle$, where $|\psi(t_i + T)\rangle = U(t_i + T, t_i) |\psi(t_i)\rangle$.

**TTSB-2:** TTSB occurs if the eigenstates of the Floquet operator $U_f = U(T, 0)$ cannot be short-range correlated.

Here, a state $|\psi\rangle$ is said to have short range correlations if, for any local operator $\phi(x)$,

$$\langle \phi(x) \phi(x') | \psi \rangle - \langle \phi(x) | \psi \rangle \langle \phi(x') | \psi \rangle \to 0 \text{ as } |x - x'| \to \infty,$$

i.e. if cluster decomposition holds. Notice that the Hamiltonian considered has a discrete time translational symmetry and not a continuous time translational symmetry.

## II. MODELS OF TTSB

### A. Spin ½ system

The EBN paper gives a many body localized system of one dimensional spin $\frac{1}{2}$ particles with a Floquet unitary

$$U_f = \exp(-it_0 H_{\text{MBL}}) \exp\left(i \frac{\pi}{2} \sum_j \sigma_j^z\right) = \exp(-it_0 H_{\text{MBL}}) \prod_j i \sigma_j^z$$

which has a period $T = t_0 + \pi/2$. The unitary has two Hamiltonians applied one after the other, the first being the $-\sum_j \sigma_j^z$ which when applied for a period $\pi/2$ effectively flips all the spins in the chain about the z-direction. The second is a many body localized Hamiltonian $H_{\text{MBL}}$ which gives an Ising interaction coupled with an external magnetic field along the z-direction with a perturbation considered along the x-direction, i.e.

$$H_{\text{MBL}} = \sum_i \left(J_i \sigma_i^+ \sigma_{i+1}^- + h_i^z \sigma_i^z + h_i^x \sigma_i^x\right)$$

where $J_i$, $h_i^z$ and $h_i^x$ are uniformly chosen from $J_i \in \left[\frac{1}{2}, \frac{3}{2}\right]$, $h_i^z \in [0, 1]$ and $h_i^x \in [0, h]$. Only slightly perturbed systems with small values of $h$ are considered.
For no perturbation (i.e. $h = 0$), this system can be analytically solved to show that TTSB occurs. Suppose we consider that $|\{s_i\} \rangle$ with $s_i = \pm 1$ are the eigenstates of the operators $\sigma^z_k$ so that $\sigma^z_k|\{s_i\} \rangle = s_k|\{s_i\} \rangle$. As $H_{\text{MBL}}$ commutes with $\sigma^z$ when $h = 0$, states $|\{s_i\} \rangle$ are also eigenstates of $H_{\text{MBL}}$ with the eigenvalues as $H|\{s_i\} \rangle = (E^+(\{s_i\}) + E^-(\{s_i\}))[\{s_i\}]$ where we define $E^+(\{s_i\}) = \sum_j s_j s_{j+1}$ and $E^-(\{s_i\}) = \sum_j h_j s_j$. With these notations the eigenstates of the Floquet operator can be written as:

$$|\Psi^\pm \{s_i\} \rangle = \exp \left( \frac{i}{2} t_0 E^- (\{s_i\}) \right) |\{s_i\} \rangle \pm \exp \left( -\frac{i}{2} t_0 E^- (\{s_i\}) \right) |\{-s_i\} \rangle$$

The corresponding Floquet eigenvalues are $\pm \exp \left( i t_0 (\{s_i\}) \right)$. Notice that the eigenstates satisfy TTSB-2. To see that TTSB-1 is satisfied as well, we study the variation of expectation values of $\sigma^z$ with time:

Let $U_1 = \exp \left( -i t_0 \sum_j \left( J_j \sigma^z_j \sigma^z_{j+1} + h_j \sigma^z_j \right) \right)$ and $U_2 = \exp \left( i \frac{\pi}{2} \sum_j \sigma^+_j \right) = \prod_j i \sigma^+_j$, so that $U_f = U_1 U_2$. Considering $|\phi \rangle$ as the initial state, we have

$$\langle \sigma^z \rangle_{t=nT} = \langle \phi | \left( U_f \right)^n \sigma^z_k (U_f)^n | \phi \rangle = \langle \phi | \left( U_2 U_1 \right)^n \sigma_k^z (U_1 U_2)^n | \phi \rangle = \langle \phi | \sigma^z_k | \phi \rangle = (-1)^n \langle \sigma^z \rangle_{t=0}$$

This shows that the expectation values of $\sigma^z$ show oscillatory behavior are periodic with period $2T$ hence verifying that TTSB-1 is satisfied.

The EBN paper goes on to argue that the above behavior is seen even when weak perturbations to the Floquet unitary such as small deviations in the length of application of the flipping Hamiltonian from $\pi/2$ and non-zero values of $h$ do not destroy TTSB. The behavior of the system under non zero values of $h$ can be studied by simulating the systems. This system is simulated by using time-evolving block decimation scheme (TEBD) [6] which is discussed in the appendix.

Fig. 1a. shows the disorder averaged expectation values of $\sigma^z$ and $\sigma^+\sigma^-$ over 100 disorders for a system size of $L=10$, $h=0.3$ and $t_0 = 1$ with the spin flips done instantaneously. The TEBD calculations were done with a Trotter step of $0.1T$ and a bond dimension of $\chi = 6$.

The lifetimes of the TTSB observed in the perturbed systems can be studied by simulating the system using exact diagonalization (ED). We define magnetization $Z(t)$ absolute value of $\langle \sigma^z \rangle$. Fig. 1b. shows the variation of the disorder averaged magnetization over 500 disorders with time for a system with
$h=0.3$, $t_o = 1$ and an initial product state polarized along the $z$-direction for small system sizes. We can see that the lifetimes diverge exponentially with the system sizes.

Fig.1a. Time dependence of disorder averaged $\langle \sigma^z \rangle$ and $\langle \sigma^x \rangle$ for a system of 10 spin $\frac{1}{2}$ sites shows that the oscillations of the former persists while that of the latter dies down.

Fig.1b. The decay of disorder averaged magnetization $Z(t)$ in spin $\frac{1}{2}$ system with time.

In this spin $\frac{1}{2}$ system, we observed that the expectation value of $\sigma^z$ had a period twice that of the Hamiltonian. In an attempt to observe TTSB with the expectation value of some operator having a period three times that of the Hamiltonian, we study a spin 1 system.
B. Spin 1 system

We consider a one-dimensional spin 1 system with a Floquet operator similar to the one considered in the spin $\frac{1}{2}$ system. We use the following notation for the operators. The matrix form of the operators are expressed by considering the standard basis.

\[
S^z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad S^x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \theta = \frac{i}{\sqrt{3}} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}
\]

\[
\Sigma^z = \exp \left( i \frac{2\pi}{3} S^z \right) = \begin{bmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad \Sigma^x = \exp \left( -i \frac{2\pi}{3} \theta \right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\]

where $\omega = e^{\frac{2\pi}{3}}$.

The Floquet unitary is

\[
U_f = \exp(-i t_0 H_{MBL}) \exp \left( -i \frac{2\pi}{3} \sum_i \theta_i \right) = \exp(-i t_0 H_{MBL}) \prod_i \Sigma_i^x
\]

Analogous to the Spin $\frac{1}{2}$ case, the Floquet unitary has two Hamiltonians applied one after the other with a total period of $T = t_0 + \frac{2\pi}{3}$. First the Hamiltonian $\sum_i \theta_i$ applied for a period $\frac{2\pi}{3}$ cycles every spin in the chain between the states with magnetization along the z-direction being 1, 0 and -1 (referred to as cycling of spins). This is in a way the analog of a spin flip in spin $\frac{1}{2}$ systems. This is followed by the Hamiltonian $H_{MBL}$ given by:

\[
H_{MBL} = \sum_i J_i \left( \left( \Sigma_i^z \right)^\dagger \Sigma_i^z + \Sigma_i^z \left( \Sigma_i^z \right)^\dagger \right) + h_i^z S_i^z + h_i^x S_i^x
\]

where $J_i, h_i^z$ and $h_i^x$ are uniformly chosen from $J_i \in \left[ \frac{1}{2}, \frac{3}{2} \right], h_i^z \in [0,1]$ and $h_i^x \in [0,h]$.

The first part of the Hamiltonian is the interaction term and the second part is that of the external magnetic field along the z-direction with a perturbation along the x-direction. This specific interaction term is chosen out of the many possibilities of the interaction terms in spin 1 systems because this interaction term is invariant under the cycling of spins which is vital for the kind of behavior we wish to have in the system.

Just as in the spin $\frac{1}{2}$ case, this system can be solved analytically for no perturbations i.e. $h=0$ and it can be proved that TTTSB occurs. Consider the eigenstates of $S_z$ as $\{s_i\}$ with $s_i = \{1,0,-1\}$ such that $S_z^z \{s_i\} = s_i \{s_i\}$. For $h=0$, we have $H_{MBL} \{s_i\} = E^+ (\{s_i\}) + E^- (\{s_i\}) \{s_i\}$ where

$E^+ (\{s_i\}) = \sum_i \left( 2 J_i \text{Re} (\omega^s_i - h_i^x) \right)$ and $E^- (\{s_i\}) = \sum_i (h_i^z s_i)$. We define the cycling operator $\odot$ with
the following set of equations:  
\[ 1 \odot 1 = 0, \ 0 \odot 1 = -1, \ -1 \odot 1 = 1 \] and \[ s_i \odot n = (s_i \odot 0) \odot (n-1) \] for \( n \in \mathbb{N} \) and \( n \neq 1 \).

Using the notation \( \alpha \in \{1,2,3\} \) and \( E(\{s_i\}) = \frac{1}{3} [E^-((s_i \odot 2)) - E^-((s_i))] \), the Floquet eigenstates are
\[ |\Psi_{\alpha} \{s_i\}\rangle = e^{\omega \alpha} \exp\left(i t_0 E(\{s_i\})\right) |\{s_i\}\rangle + e^{2\omega \alpha} \exp\left(i t_0 E((s_i \odot 1))\right) |\{s_i \odot 1\}\rangle + e^{3\omega \alpha} \exp\left(i t_0 E((s_i \odot 2))\right) |\{s_i \odot 2\}\rangle \]
with the eigenvalues \( e^{\omega \alpha} \exp\left(i t_0 E^+(\{s_i\})\right) \). These eigenstates satisfy TTSB-2. The variation of expectation values of \( S_z \) with time can be found as follows:

Let \( U_1 = \exp\left(-it_0 \sum J_i \left( \sum_{i=1}^\alpha \Sigma_i + \sum_{i=1}^{\alpha+1} \Sigma_i \right) \right) \) and \( U_2 = \exp\left(-i \frac{2\pi}{3} \sum \varphi_i \right) \), so that \( U_f = U_i U_2 \). Considering \( |\varphi\rangle \) as the initial state, we have
\[
\langle \Sigma_{k,l} \rangle_{t+nT} = \langle \varphi | (U_f^n)^\dagger \Sigma_{k,l} (U_f^n)^n | \varphi \rangle = \langle \varphi | (U_2^n U_1^n)^n \Sigma_{k,l} (U_1 U_2^n)^n | \varphi \rangle = \omega^n \langle \varphi | \Sigma_{k,l} | \varphi \rangle \]
\[ = \omega^n \langle \varphi | \Sigma_{k,l} | \varphi \rangle \]
where \( U_1, \Sigma_{k,l} = 0 \) and \( \Sigma_{k,l} = 0 \) for \( k \neq k \)
\[ \Sigma_{k,l} = \omega \Sigma_{k,l} \]
\[ = \omega^n \langle \varphi | \Sigma_{k,l} | \varphi \rangle \]
\[ = \omega^n \langle \varphi | \Sigma_{k,l} | \varphi \rangle \]

Since \( S_z = \frac{1}{i\sqrt{3}} \left( \Sigma_{k,l} - i \left( \Sigma_{k,l} \right) \right) \), we get the result
\[ \langle S_z \rangle_{t=nT} = \frac{2}{\sqrt{3}} \text{Im} \left( \omega^n \langle \varphi | \Sigma_{k,l} | \varphi \rangle \right) \]
which shows that the expectation values of \( S_z \) are periodic with a period thrice that of the Hamiltonian. Hence TTSB-1 is satisfied for this system.

When perturbations to the system are considered, the behavior of the system may be studied by simulations. The system is simulated by using TEBD. Fig.2a. shows the disorder averaged expectation values of \( S_z \) and \( S_x \) over a 67 disorders for a system size of \( L=50 \), \( h=0.3 \) and \( t_0 = 1 \) with the cycling of spins done instantaneously. The TEBD calculations were done with a Trotter step of 0.1T and a bond dimension of \( \chi = 20 \). We see that the TTSB behavior is still seen in the system. To study the lifetime of TTSB in the system, we simulate it using ED. Here we define magnetization \( Z(t) \) as the maximum value of the absolute value of \( \langle S_z \rangle \) in the time interval \( t \) to \( t+2T \). Fig.2b. shows the variation of the disorder averaged magnetization over 500 disorders, with time for a system with \( h=0.3 \), \( t_0 = 1 \) and an initial product state polarized along the z-direction for small system sizes. Notice that just as in the case of the spin \( \frac{1}{2} \) systems, the lifetimes diverge exponentially with the system size but the decay of magnetization is not as steep.
Fig. 2a. Time dependence of disorder averaged $\langle S^z \rangle$ and $\langle S^x \rangle$ for a system of 50 spin 1 sites.

Fig. 2b. The decay of the disorder averaged magnetization $Z(t)$ in spin 1 system with time.
C. Spin S system

The TTSB model of spin 1 system proposed can be naturally extended to spin S systems. We use the following notations.

\[
S^z = \begin{bmatrix}
S \\
S-1 & 0 \\
& \ddots \\
0 & \ddots & S \\
& & -S
\end{bmatrix} \quad \quad \quad \quad S^x = \begin{bmatrix}
0 & \sqrt{2S} \\
\sqrt{2S} & 0 \\
& \ddots & \ddots \\
& & \sqrt{2(2S-1)} & 0 \\
& & & \ddots & \ddots \\
& & & & \ddots & \sqrt{2S} \\
& & & & & 0
\end{bmatrix}
\]

\[
\Sigma^z = \begin{bmatrix}
1 \\
\xi & 0 \\
& \ddots \\
0 & \ddots & \xi^{-d-1}
\end{bmatrix} \quad \quad \quad \quad \Sigma^x = \begin{bmatrix}
0 & 1 \\
0 & 0 & 1 \\
& \ddots & \ddots \\
& & 1 & 0 & 0 \\
& & & 1 & 0 & 0
\end{bmatrix}
\]

where \( d = 2S + 1 \) and \( \xi = e^{\frac{2\pi i}{d}} \)

The Floquet unitary and the many body localised Hamiltonian considered are similar in structure to that considered in the spin 1 case.

\[
U_f = \exp(-it_o H_{MBL}) \prod_i \Sigma_i^x
\]

where \( H_{MBL} = \sum_j J_j \left( (\Sigma_{j}^z)^{\dagger} \Sigma_{j+1}^z + \Sigma_{j}^z (\Sigma_{j+1}^z)^{\dagger} \right) + h_j S_j^z + h^* S_j^x
\]

To show that the system is in fact a model of TTSB, we find the variation of expectation value of \( S_z \) with time.

\[
S^z = \frac{1}{d} \sum_{a,b=1}^d \left( (S+1-b) \xi^{-a(b-1)} (\Sigma_z^a)^{\dagger} \right) \quad \text{and}
\]

\[
\left\langle \left( \Sigma_k^z \right)^{\dagger} \right\rangle_{t=0} = \langle \phi | (U_{f})^a (\Sigma_i^z)^{a} (U_{f})^a | \phi \rangle = \langle \phi | (U_{2} U_{1})^a (\Sigma_i^z)^{a} (U_{2})^a | \phi \rangle
\]

\[
= \langle \phi | (\Sigma_k^z)^a (\Sigma_i^z)^{a} (\Sigma_k^z)^{a} | \phi \rangle \quad \text{as} \quad \left[ U_{1}, (\Sigma_i^z)^{a} \right] = 0 \quad \text{and} \quad \left[ (\Sigma_i^z)^{a}, \Sigma_k^z \right] = 0 \quad \text{for} \quad k \neq k
\]

\[
= \xi^{-a} \langle \phi | (\Sigma_i^z)^{a} | \phi \rangle \quad \text{as} \quad \left( \Sigma_k^z \right)^a \Sigma_i^z = \xi^{-a} \Sigma_i^z \left( \Sigma_k^z \right)^a
\]

Hence we have

\[
\left\langle S_k^z \right\rangle_{t=0} = \frac{1}{d} \sum_{a,b=1}^d \left( (S+1-b) \xi^{-a(b-1)+a} \left\langle (\Sigma_z^a)^{\dagger} \right\rangle_{t=0} \right)
\]
Here again we can see that the expectation values of $S_z$ are periodic (by substituting $n=d$) with a period $d$ times the period of the Hamiltonian hence satisfying TTSB-1.

Fig.2. shows the expectation values of $S^z$ and $S^x$ for a system with $S=3/2$ and $S=2$ respectively and having a system size of $L=50$, $h=0$ and $t_0=1$ with the cycling of spins done instantaneously. The TEBD calculations were done with a Trotter step of 0.1$T$ and a bond dimension of $\chi = 4$ and $\chi = 5$ respectively.

Fig.3. Time dependence of $\langle S^z \rangle$ and $\langle S^x \rangle$ for a system of 50 spin $3/2$ (top panel) and spin 2 (bottom panel) sites with no perturbations.
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Appendix: Time Evolving Block Decimation (TEBD)

TEBD is an algorithm used to efficiently simulate the time evolution of quantum systems which have low long range entanglement throughout the period for which it is evolved. Generally low energy states of quantum systems follow area law of entanglement entropy (i.e. the entanglement entropy scales linearly with the size of boundary of the system) \[7\] while higher energy states follow the volume law of entanglement. But in many body localised systems, even higher energy states follow area law of entanglement which implies that all states in a many body localised system have only short range correlations. This allows for many body localised systems to be efficiently simulated by TEBD. Vidal has given a comprehensive method of implementing TEBD \[6\].

The initial state \(|\Psi\rangle\) expressed in the matrix product state (MPS) form is:

\[
|\Psi\rangle = \sum_{i_1,...,i_n} \Gamma_{a_1^{[1]} a_1^{[2]} \cdots a_1^{[n]}} |i_1...i_n\rangle
\]

where \(i \in \{1,2,...,2S+1\}\) and \(|i_1...i_n\rangle\) represents the spin \(S\) orthonormal basis of a \(n\) spin system.

For short-range correlated states, the components of \(\lambda\) vectors fall exponentially \[5\]. Exploiting this fact can lead to significant decrease in the storage space and provide a speed up in the calculations required to simulate these systems. The number of significant values of \(\lambda\) considered is taken to be \(\chi\). For simulating the system with zero error, the value of \(\chi\) that needs to be considered is the Schmidt rank of the system.

The evolution of the system having Hamiltonians with single spin interaction or neighboring spin interactions is relatively simple in this method. Single spin interactions are carried out by updating just the appropriate \(\Gamma\) tensor. Considering \(U\) as the unitary corresponding to the Hamiltonian acting on the \(l\)th spin, the updated \(\Gamma\) tensor is given by \(\Gamma_{ij} = U \Gamma_{ij} U\). The computational cost for the operations involved is \(\Theta(\chi^2)\) basic operations.

The evolution of the system with neighboring spin interactions can be carried out by updating the corresponding two \(\Gamma\) tensors and the corresponding \(\lambda\) vector. The computational cost for carrying out all the calculations involved is \(\Theta(\chi^3)\).
Hence using TEBD, a computation whose complexity would have scaled exponentially with the number
of spins if done conventionally, can be done with a complexity of polynomial of $\chi$ which in turn grows
as a polynomial of the number of spins if there is low long range entanglement.

References