Examining Parameter Dependent Chaos Properties of Rikitake's Simplified Geodynamo

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Abstract

The primary goal of our research was to take a second look at simplified models of Earth's magnetic field -- in particular, one that displays chaotic reversals. We analytically and numerically analyzed a nondimensionalized version of the set of coupled ordinary differential equations governing Rikitake's simplified double dynamo model for the Earth's magnetic field. Additionally, we created phase diagrams similar to those of Kono's to discover new peculiarities in the behavior of solutions' dependence on parameters. The result of which demonstrates the unpredictable behavior of even simple models of the geomagnetic reversals.

Introduction

Any information that we have gleaned about the history of the geomagnetic field to any considerable time scale comes from evidence located inside the earth's crust. Such evidence indicates a chaotic history of dipole dominated magnetic fields with aperiodic reversals of polarity (Glatzmaier and Coe 2007). It was our primary concern to analyze a simplified model of such reversals rooted in electromechanical dynamics.

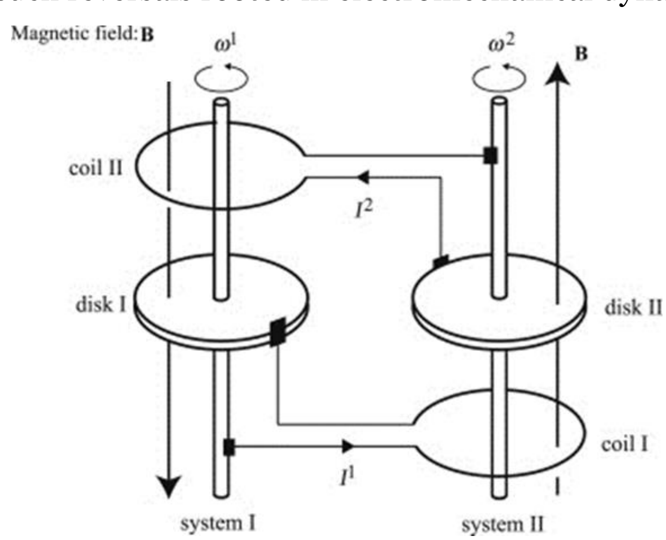

Figure 1 - Rikitake Dynamo (Yajima and Nagahama 2009)

Finding a mathematical model that accurately describes the behavior of Earth's magnetic field is a high priority among many geological scientists. A full analysis using magnetohydrodynamics provides a fairly complete approach. The downside to this approach is laborious numerical calculations coupled with a persistent need for simplifying assumptions of the conditions in the outer core. It was with these considerations in mind that we decided instead to analyze Rikitake's Double Dynamo, a simplified yet effective model of Earth's magnetic field in that it displays the approximate feedback, electromechanical, and reversal properties of the Earth's magnetic field (Rikitake 1958). Figure 1 shows the schematics of Rikitake's Dynamo. Although not explicitly stated in the equations, the magnetic field strength is proportional to the current running through each loop (proportional to the sum of the two currents in the simplifying case that the loops are along the same axis, but are noninteracting). The direction of the field is determined by the sign of the sum of the currents. The governing equations are as follows.

$$L\frac{dI_1}{dt} = M\Omega_1 I_2 - RI_1 \qquad (1)$$

$$L\frac{dI_2}{dt} = M\Omega_2 I_1 - RI_2 \qquad (2)$$

$$C\frac{d\Omega_1}{dt} = G - MI_1 I_2 \qquad (3)$$

$$C\frac{d\Omega_1}{dt} = G - MI_1 I_2 \qquad (4)$$

I_1, I_2, Ω_1, and Ω_2 are the currents and angular velocities of the two loops and two rotating disks respectively and also the dependent variables. The current being of particular interest as alluded to earlier. L is the self-inductance of the loops, R is the resistance of the loops, M is the mutual inductance of the loops, G is the driving torque, and C is the moment of inertia of the disk about its rotation axis. Note that all of these values were assumed to be the same across the two coupled systems.

It isn't enough for a mathematical model to fit certain physical behaviors though. There should also be some sort of physical inspiration for the model. Here, we simply take charged ions moving through conductive packets of molten metal driven by convection and Coriolis forces in complex ways and reduce that to two rotating disks attached to loops of wire that interact in an electromagnetic feedback system.

It was our goal to perform a parameter dependent analysis of solutions to this particular set of ODEs in order to get a deeper insight into the behavior of this seemingly simplistic system. Previous analysis of Rikitake's model has been performed fairly extensively in the past. Hoshi and Kono used computational methods in order to create a phase diagram of solutions.

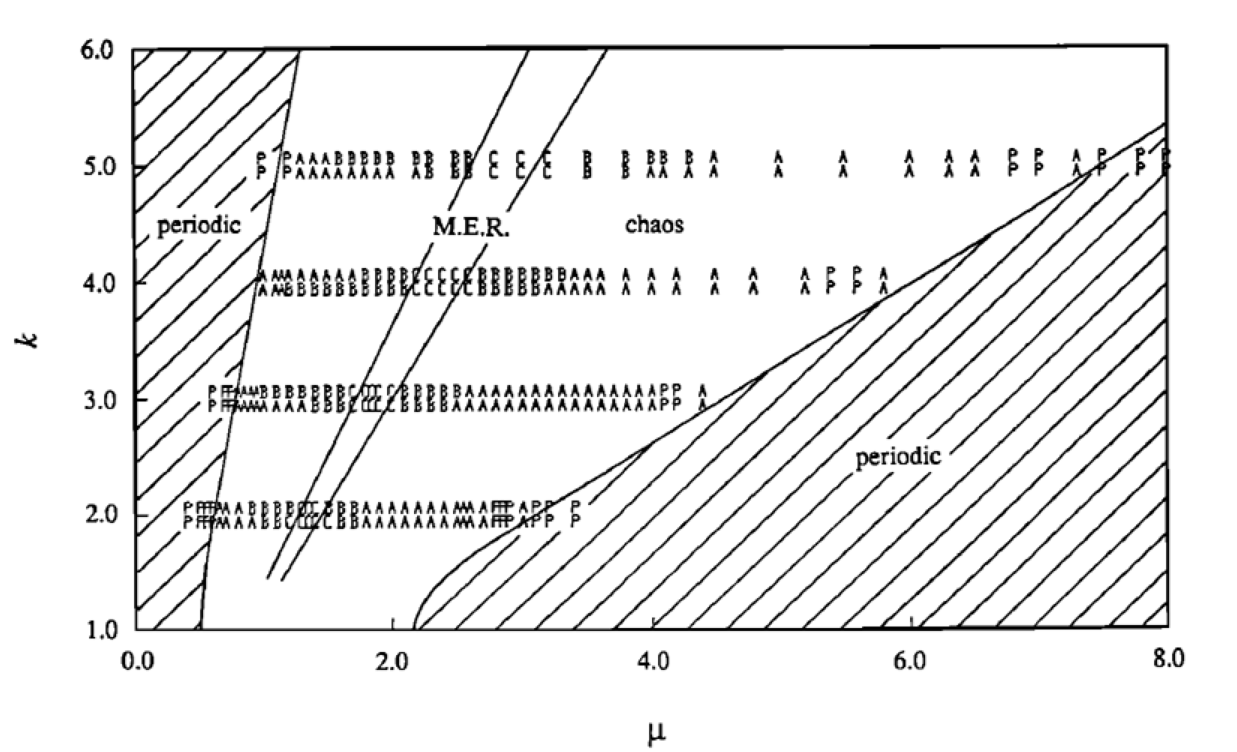

Figure 2 - Phase Diagram (Hoshi and Kono 1988)

We endeavored to create a higher resolution mapping of solutions. In doing so, we managed to discover some new previously undiscovered phenomena in solutions of certain parameterization.

Analytical Methods

The logical first step to analyzing parameter dependence of a set of differential equations of this form is to attempt to create a nondimensionalization. For the numerical portion of the research, we borrowed Cook and Robert's nondimensionalization.

$$\dot{X}_1 + \mu\dot{X}_1 = YX_2 \tag{5}$$
$$\dot{X}_2 + \mu\dot{X}_2 = (Y - A)X_1 \tag{6}$$
$$Y = 1 - X_1X_2 \tag{7}$$

Note that the system of four differential equations is reduced to that of three with the simple observation that the right-hand side of (3) equals the right hand side of (4). The implication being that the separation between disk velocities is constant. A is therefore the difference between the angular velocities of the first and second disks. The current is measured in units of the steady state current of (3) and (4). The time is measured in the geometric mean of the natural electric and mechanical time scales. μ is a parameter that is equal to the square root of the ratio of the mechanical and electric time scales.

When we let

$$A = \mu(K^2 - K^{-2}), \tag{8}$$

The steady state solutions are $(\pm K, \pm K^{-1}, \mu K^2, \mu K^{-2})$ for $K \geq 1$ where the plus/minuses are taken together. Since A is entirely determined by$K$, a phase diagram can be made that maps the K-μ dependence of solutions (see figure 2).

Linearizing about the steady states yields eigenvalues of

$$\lambda = -2\mu, +i\sqrt{K^2 + K^{-2}}, -i\sqrt{K^2 + K^{-2}}. \tag{9}$$

Two purely imaginary eigenvalues betray little information about solutions – implying an oscillatory nature. The negative real value implies some stability for solutions along the associated eigenvector.

Numerical Methods

In order to create our own phase diagram, we needed to generate solutions to our ODEs. Due to nonlinearity inherent in each of equations (1)-(4), analytical solutions are difficult if not impossible to come by. Linearizing about the steady state solution only gives so much insight into the behavior of solutions. The natural solution to this problem is a numerical approach.

We implemented a 4th order Runge Kutta ODE approximation method using MATLAB 2010a. An array of μ and K values are taken as inputs and the program saves images of the plots of the solutions in X_1-X_2 phase space. The initial conditions being set sufficiently far away from the steady state solutions and the solutions truncated as to give a good idea of the long term behavior of solutions. We solved the initial value problems up to 500 time-scale units, truncating solutions to the last 200 units. We then looked through the individual images and classified the solutions broadly as chaotic or periodic.

Results and Discussion

The highest precision of parameter adjustment we did was a range of K from 1 to 6 and a range of μ from 0 to 8 – just like Hoshi and Kono's phase diagram. Some of the typical solutions found are listed below followed by our phase diagram.

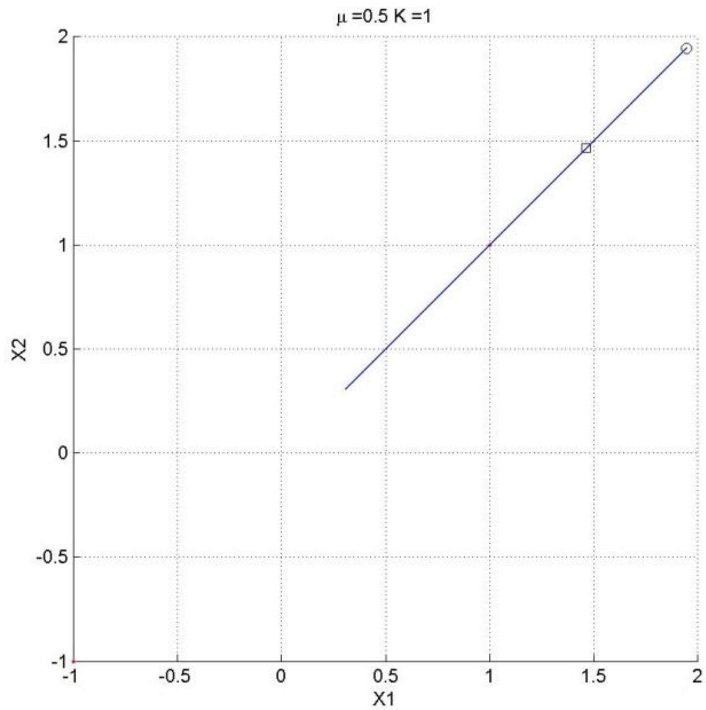

Figure 3 – Periodic Solution without Reversing

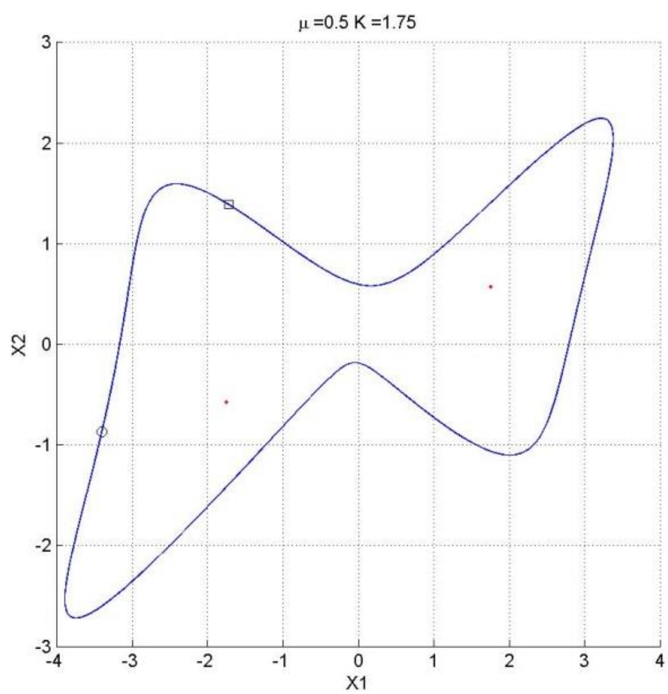

Figure 4 - Periodic Solution with Reversing

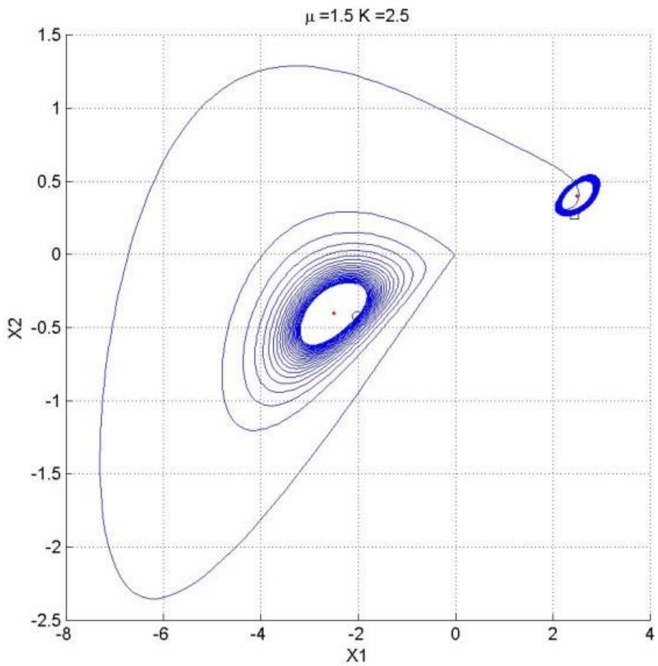

Figure 5 - Minimum Entropy Solution

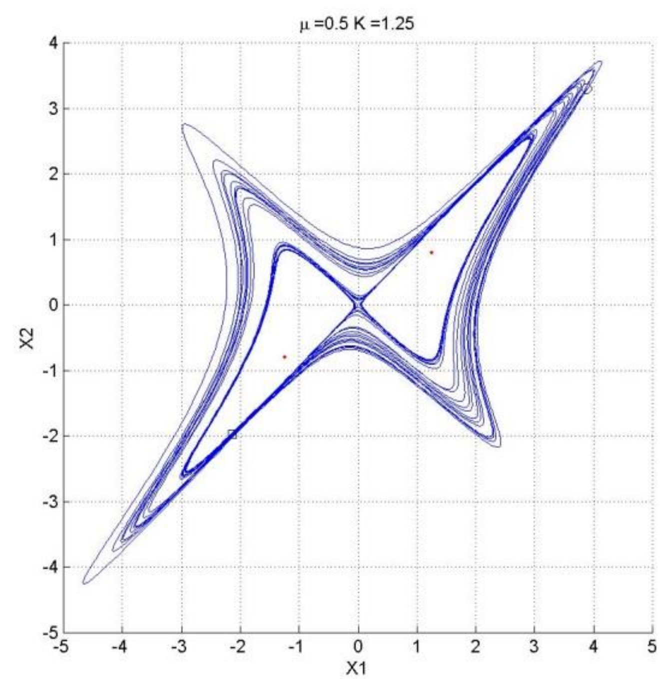

Figure 6 - Chaotic Solution

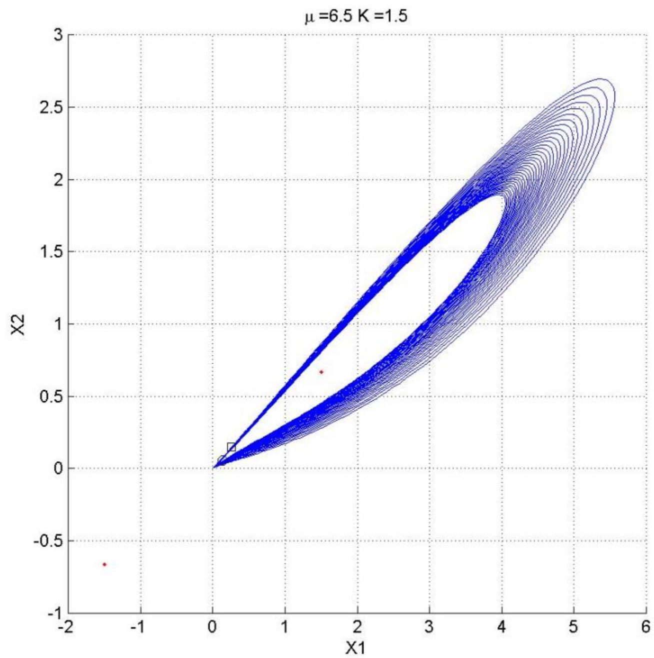

Figure 7 - Ambiguous Solution

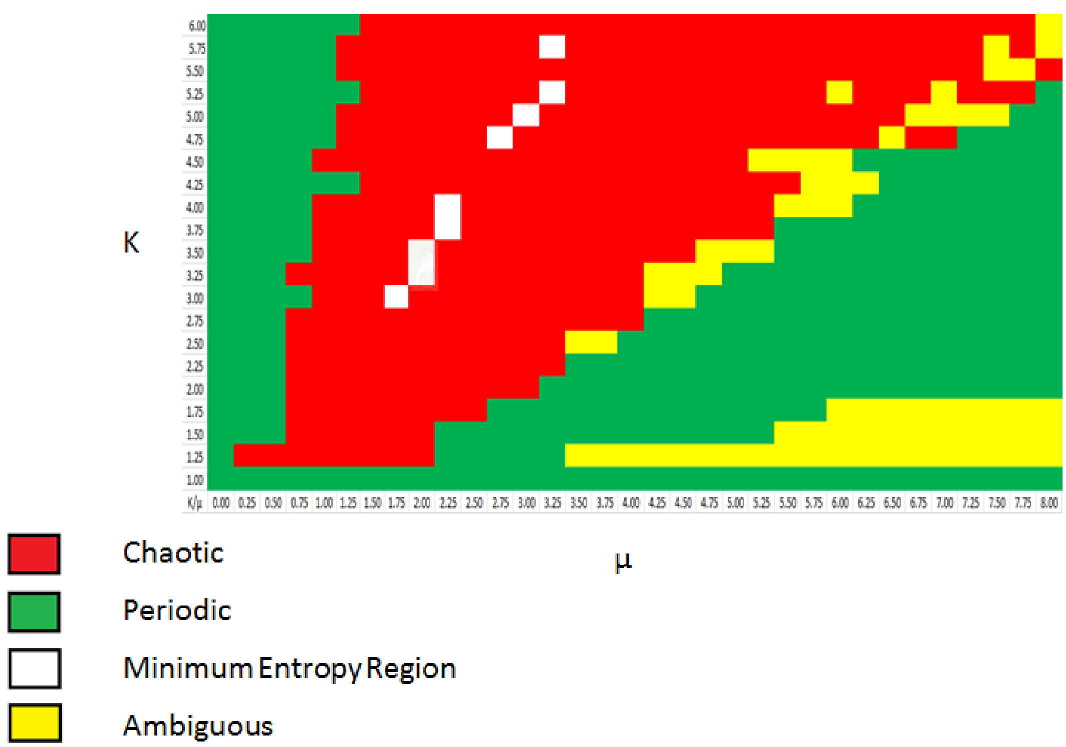

Figure 8 - Our Phase Diagram

In terms of solutions that give a general idea of the characteristics of geomagnetic field fluctuations, the chaotic solutions are the most accurate. The following is a characteristic plot of the sums of current versus time for one such chaotic solution.

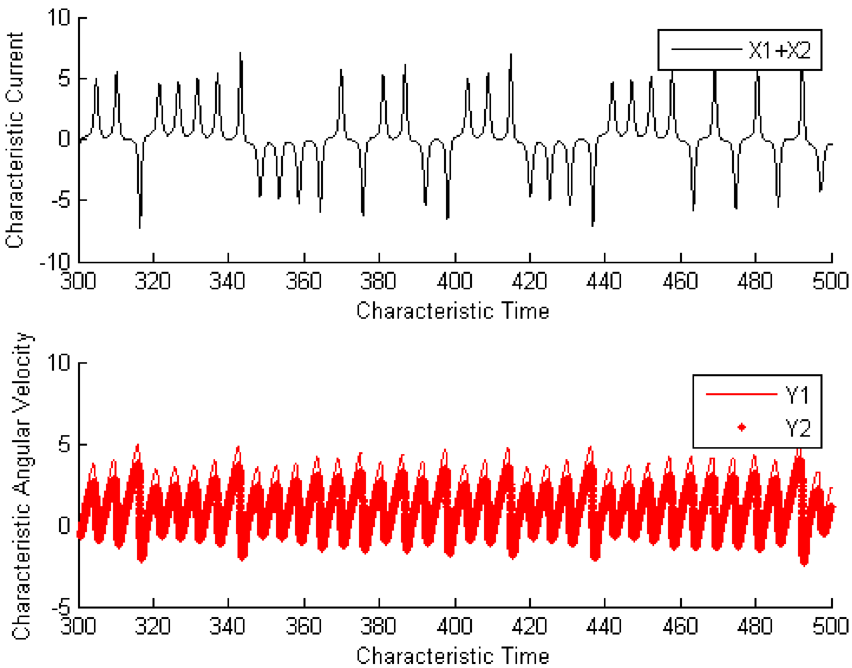

Figure 9 - Chaotic Solution Evolution in Time

Interestingly, there are clear oscillations in between reversals which are indeed another characteristic of Earth's magnetic field. To claim that this model accurately describes such oscillations is a stretch since the amplitude of such oscillations is far too great in this model. The model does however nicely capture the chaotic nature of reversals.

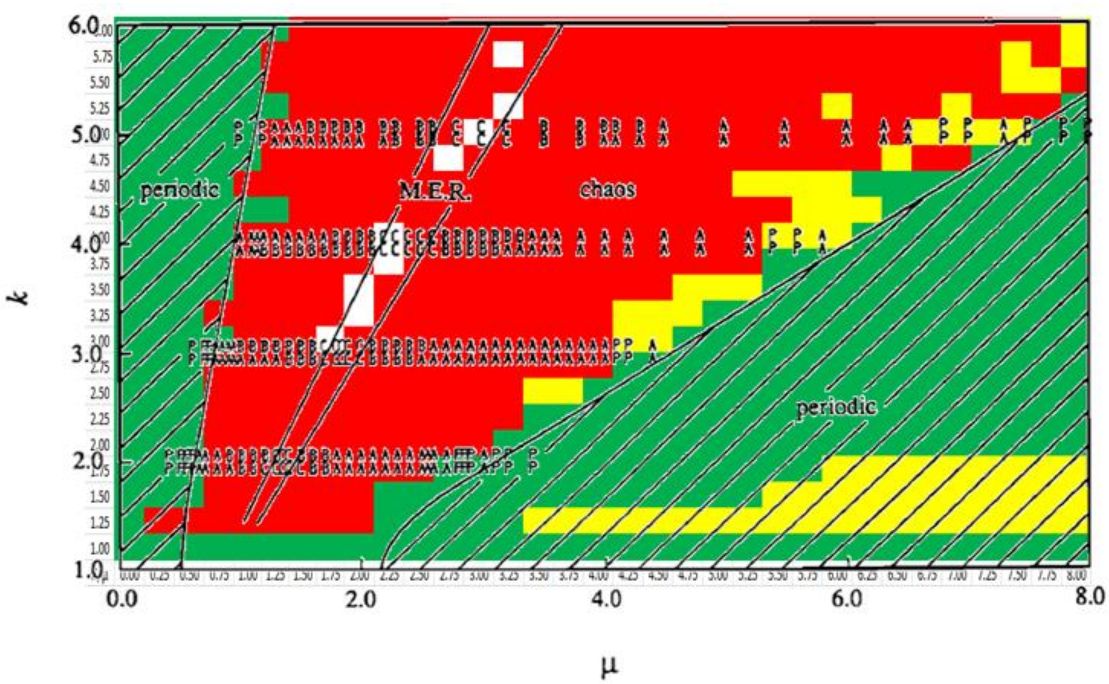

Figure 10 - Overlaid Phase Diagrams

Overlaying the two phase diagrams (Figure 10) shows a general agreement between our results and Hoshi and Kono's. Two observations of interest are to be had here however. Firstly, it seems as if the boarders between periodic and chaotic solutions in parameter space aren't quite as smooth as previously thought. The perhaps more interesting observation is that the lower right hand corner has a region of chaotic solutions unmarked by Hoshi and Kono. See Figure 7 for a typical example. Unlike other chaotic solutions, solutions in this ambiguous region do not display reversals of current. This indicates that despite being chaotic, they're probably not very good parameters for an accurate model of the geodynamo. The main significance in these two observations is that they indicate that even a very simplified model of the geomagnetic field can have unexpectedly complicated bifurcation curves in parameter space.

Further Research

Looking to the future, we are interested in further exploring the phase space of this particular set of ODEs. Specifically, we think that the bifurcation curves should be examined further. By upping the resolution and localizing the analyzed portions to those of specific interest, computation time can be reduced while useful data output increases. Additionally, further research into the non-reversing chaotic region should be done.

$$\dot{X}_1 = X_2 Y - X_1 \tag{8}$$

$$\dot{X}_2 = X_1(Y - A) - X_2 \tag{9}$$

$$\dot{Y} = \mu(1 - X_1 X_2) \tag{10}$$

$$\mu = \frac{LGM}{CR^2} \tag{11}$$

$$A = K^2 - K^{-2} \tag{12}$$

$$I/X = \sqrt{\frac{G}{M}} \tag{13}$$

$$\Omega/Y = \tau/t = \frac{R}{M} \tag{14}$$

$$\text{Steady state solution} = (\pm K, \pm K^{-1}, K^2, K^{-2}) \tag{15}$$

During the duration of our research, we developed a novel nondimensionalization of Rikitake's governing ODEs. The governing equations are equations (8)-(10). Aesthetically different from (5)-(7) in that only one equation has the parameter μ instead of two. Another aesthetic difference is that the steady state solutions are now parameterized by K alone. The motivation behind this nondimensionalization is making current and angular velocity both measured in characteristic units. Time is measured in the inverse unit of characteristic angular velocity which is logical in the mechanical sense at the very least. μ is perhaps less significant in this nondimensionalization though since it has no known inherent physical significance like Robert and Cook's did. Little numerical analysis has been performed with this particular nondimensionalization though. Perhaps analysis will provide a smoother more intuitive phase diagram. Alternatively, since A and K are dependent on each other in both nondimensionalizations, additional phase diagrams can be created by examining A-μ parameter space instead. Such a novel approach may yield more information about the parameter dependence of solutions to the Rikitake Dynamo equations and the type of solutions.

<u>Sources</u>

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