UTRIP 2013: Novel Phase Transitions in a Damped, Driven Cavity System

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1 Introduction

Interaction between light and matter plays a central role in Cavity Quantum Electrodynamical (QED) systems. Schematically, a cavity system consists of standing photon modes confined inside an optical cavity. The cavity also contains, most generally, a collection of finite-level systems that are coupled to a thermal reservoir. The system can also be driven by a driving photon field.

In the laboratory, this setup can be realized in different ways, such as circuit QED [8], semiconductors [5] or cavity-confined cold atoms [1]. The driving field is usually achieved using a laser. Theoretically, the cavity system can be modelled using the Dicke Model Hamiltonian [4], which takes into account the atom-photon, atom-bath and photon-bath interactions. Other features, such as atom-atom interaction [3] can also be included in the model. It has been shown theoretically [9] and experimentally [1] that the photon mode can induce interesting states of matter in the strong coupling regime [3;1].

The system that we consider is that of a large collection of 2-level atoms confined in a cold cavity. Conventionally, the Rotating Wave Approximation (RWA) is applied to the Dicke model (giving the Tavis-Cummings model [10]), which greatly simplifies the analysis. However, it has been shown that the RWA is not valid in the strong coupling regime (see Shirai et al [9]), which is where the system undergoes interesting transitions away from thermal equilibrium. The paper [9] also discusses a Quantum statistical approach that does not involve RWA. The results from this agree well with the Dicke transition, and also predicts other regimes in which the ground state is away from thermal equilibrium. In this project, the model was tested numerically in order to check the validity of the result away from the thermodynamic limit (finite number of atoms N).

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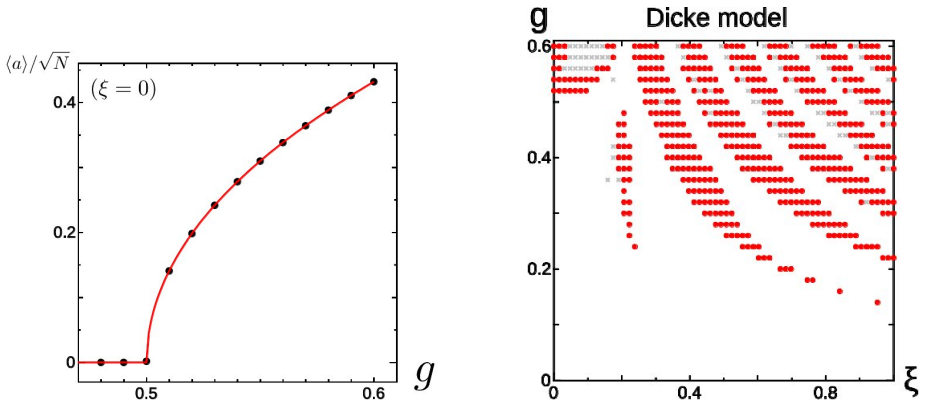

Figure 1: Left: Dicke Transition. The steady-state expectation value of the ladder operator has been plotted against the interaction strength. Classically, It is expected that the photon mode decays to 0 in the long-term. However, beyond $g = 0.5$ we see a sudden non-zero expectation value which indicates the emergence of a "superradiant" steady state. Right: When the RWA is not applied, novel phases appear at certain values (clustered in bands) of g, ξ. This can be seen as an interference-like effect between the driving field and the cavity mode. Source: Shirai et al[9]

Cavity QED holds promise for the future of quantum computation, since it is a system which exhibits useful quantum effects, such as entanglement[2], long-term coherence and interesting collective behaviour. It is also realizable in the laboratory. This problem is therefore interesting and also applicable in real-world systems.

2 Theoretical Background

Due to the large number of atoms in the system, the state inhabits a large Hilbert space, and a complete description of the quantum evolution of the system would involve an intractible number of coupled equations. Thus, we use a Quantum Statistical Mechanics approach.

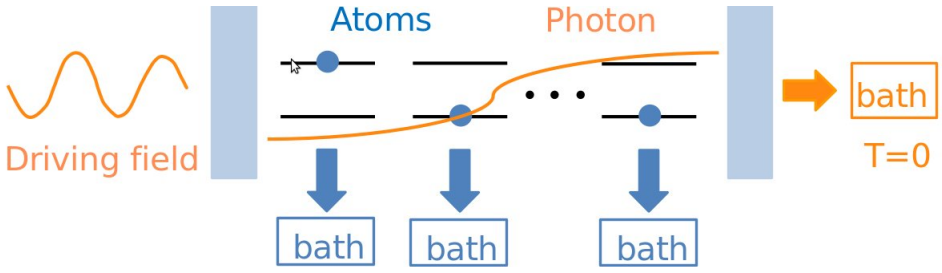

Figure 2: A schematic of the Optical cavity system.

2.1 Dicke Model

The system's Hamiltonian is given by the Dicke model[4],

$$H_S = \underbrace{\omega_p a^\dagger a}_{\text{photon}} + \underbrace{\omega_a \sum_{j=1}^{N} S_j^z}_{\text{atoms}} + \underbrace{\frac{2g}{\sqrt{N}} \sum_{j=1}^{N} S_j^x \left(a + a^\dagger\right)}_{\text{photon-atom}} + \underbrace{2\sqrt{N}\xi \cos\left(w_e t\right)\left(a + a^\dagger\right)}_{\text{driving}} \tag{1}$$

where g is the atom-photon interaction strength, ξ is the driving interaction strength and $\omega_p, \omega_a, \omega_e$ refer to the frequency of the photon, atom, and the energy gap of the atom. We take $\hbar = 1$ everywhere, and standard notation is used for the ladder operators and spin operators. We work in the long-wavelength regime, so that the atomic ensemble is essentially point-like, and we can make the simplification $e^{i\mathbf{k}\cdot\mathbf{r}} \simeq 1$.

The spin operators appear when we treat the two-state atom as (effectively) a particle with spin-$\frac{1}{2}$ in a z-directed magnetic field. The atom-photon interaction is classically motivated from the expression for the energy of a dipole in an electric field, $H = \mu \cdot \mathbf{E}$. The electric field operator due in the photon field is proportional to $a + a^\dagger$ and the dipole moment operator is proportional to $S^+ + S^- \propto S^x$. The driving field is treated semi-classically and the atom-driving interaction follows from the classical expression much like the atom-photon interaction.

2.2 Thermal Bath

In a real-world scenario, each component of the system (photon, atoms) interacts with the surroundings. The environment can be treated as a thermal reservoir, which means that it has a continuum of energy states that couple with the system. We consider two types of bath operators in our model, those that couple with the system as a whole, and those that couple to subsystems within the cavity system. Denote the bath operators using the letter A. Then, the bath Hamiltonian can be written as:

$$H_B = \sum_{j=1}^{N} \sum_{\alpha} \omega_{j,\alpha} A_{j,\alpha}^\dagger A_{j,\alpha} + \sum_{\alpha} \omega_{G,\alpha} A_{G,\alpha}^\dagger A_{G,\alpha} \tag{2}$$

The interaction between the bath and the system can be expressed in the following manner:

$$H_I = \sum_{j=1}^{N} \sum_{\alpha} \left(k_{j,\alpha} A_{j,\alpha} + k_{j,\alpha}^* A_{j,\alpha}^\dagger\right)\left(S_j^+ + S_j^-\right) + \sum_{\alpha} \left(k_{G,\alpha} A_{G,\alpha} + k_{G,\alpha}^* A_{G,\alpha}^\dagger\right) \sum_{j=1}^{N} \left(S_j^+ + S_j^-\right) \tag{3}$$

The complete Hamiltonian is then given by the sum of the above components:

$$H_T = H_S + H_B + \lambda H_I \tag{4}$$

2.3 Density Matrix Formulation

Due to the large number of interacting subsystems, we need a method that coarse-grains over microscopic behaviour and allows us to look at the aggregate behaviour. This is a justified step since all properties that can be measured in a laboratory are of a macroscopic nature: net dipole moment, average photon occupation number etc. Therefore we reformulate the problem in terms of the density matrix, which gives a probability distribution over the different microstates of the ensemble:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

To further simplify the analysis, we assume a Mean-Field (MF) form for the density matrix,

$$\rho = \rho_p \otimes \rho_a^{(1)} \otimes \rho_a^{(2)} \cdots \otimes \rho_a^{(N)} \tag{5}$$

Although this product-state form is special, T. Mori has shown[7] that in the thermodynamic limit ($N \to \infty$) this approximation is justified.

2.4 Quantum Master Equation

The Quantum Master Equation determines the evolution of the density matrix of a system. While each microstate evolves according to the Schrödinger equation, the density matrix has the following "equation of motion":

$$\frac{\partial \rho}{\partial t} = -i[H_T, \rho] \tag{6}$$

3 Equations of Motion

We work in the Schrödinger picture, in which the states (and the density matrix) evolve, whereas the operators are stationary in time. Therefore, the time evolution of the expectation value of any operator O is given by

$$\langle \dot{O} \rangle = \frac{d}{dt} \mathrm{Tr}\left(O\rho\right) = \mathrm{Tr}\left(O \frac{d\rho}{dt}\right) \tag{7}$$

With this approach, we derive the equations of motion of all the relevant quantities: $\alpha, m_x = \langle \bar{S}^x \rangle, m_y = \langle \bar{S}^y \rangle, m_z = \langle \bar{S}^z \rangle$, where $\alpha = \frac{\langle a \rangle}{\sqrt{N}}$. We have the following equations:

$$\dot{\alpha} = \left[-\kappa - i\omega_p\right] \left[\alpha + \frac{2g}{\omega_p} m_x + \frac{2\xi}{\omega_p} \cos\left(\omega_e t\right)\right] \tag{8}$$

$$\dot{m}_x = -\omega_a m_y - \gamma \left[\left(1 + \sin^2\theta\right) m_x + \left(\sin\theta \cos\theta\right) m_z + \sin\theta\right] \tag{9}$$

$$\dot{m}_y = \omega_a m_x - 2g\left[\alpha + \alpha^*\right] m_z - \gamma m_y \tag{10}$$

$$\dot{m}_z = 2g\left[\alpha + \alpha^*\right] m_y - \gamma \left[\cos\theta \sin\theta m_x + \left(1 + \cos^2\theta\right) m_z + \cos\theta\right] \tag{11}$$

These equations were solved numerically using a Runge-Kutta code that I wrote. I was able to verify the Dicke transition as well as the novel phases.

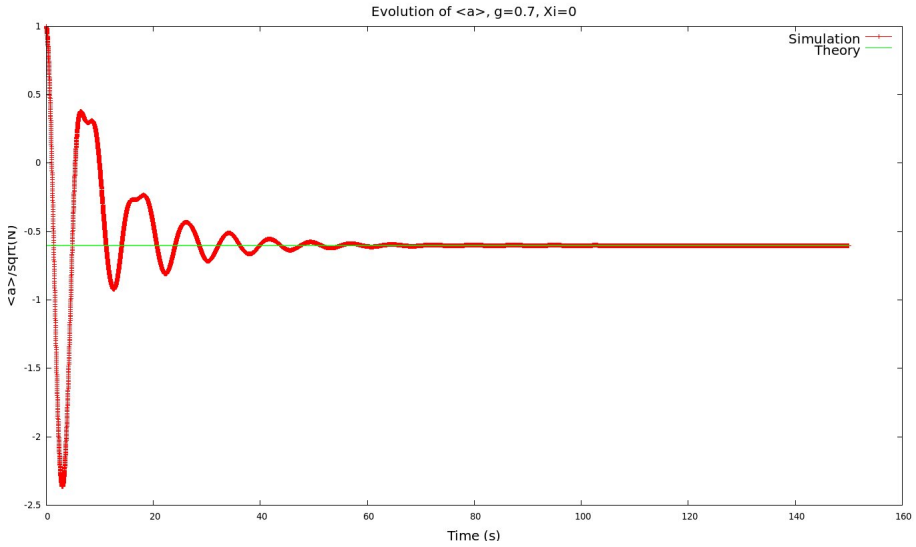

Figure 3: Time evolution of α in the super-radiant regime (at $\xi = 0$ – no driving). The steady-state value matches that expected from theory (green).

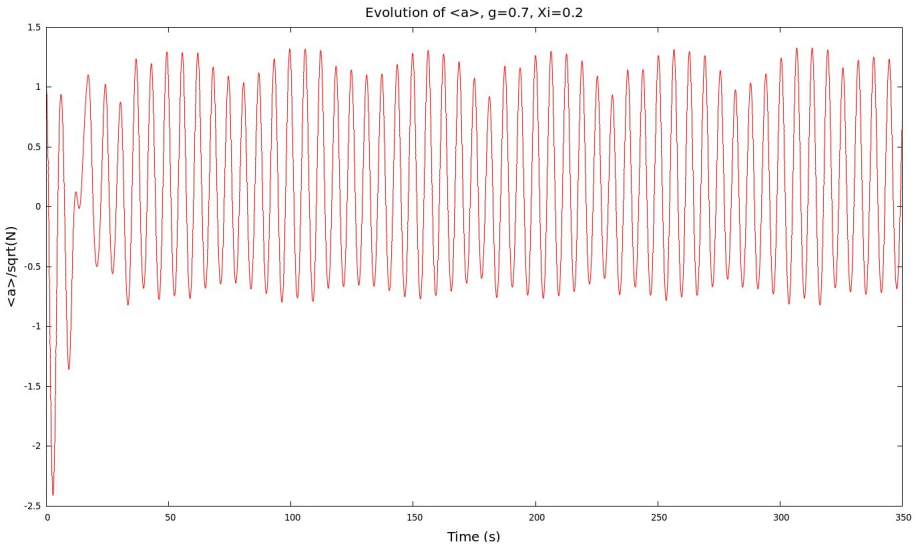

Figure 4: Novel Phase at $g = 0.7, \xi = 0.2$.

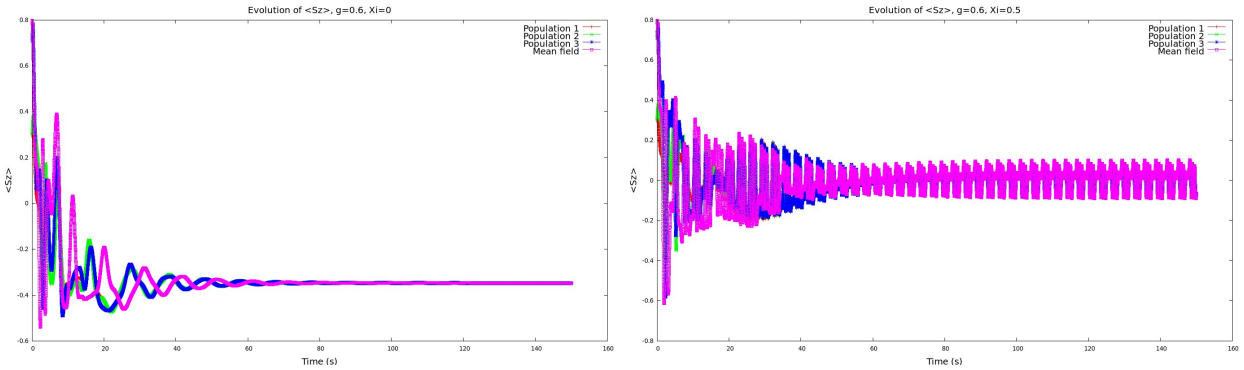

Figure 5: In these plots, different atoms start at three different initial conditions. Thus there are 3 separate populations of atoms that may evolve differently. However, for all initial conditions tried so far their behaviour appears to converge at large t. The convergent value is what one expects if there were only one population.

4 Conclusion

The Quantum Statistical approach has wide applicability in quantum dynamics problems. Focusing on the density matrix rather than the state of the system makes calculations much more tractable without losing important physics. We see that care must be taken when making approximations, as they may not be valid in the regime of interest. For example, the RWA does not yield accurate results for the Dicke transition. The MF approximation, however, is applicable as long as one considers the thermodynamic limit. Similarly, when the photon wavelength is comparable to the system size, one must include the phase $e^{i\mathbf{k}\cdot\mathbf{r}}$ in the atom-photon interaction that was omitted for this calculation.

Due to the simplifications in the analysis, a fairly simple code can be implemented to study new and exciting collective phenomena. Some of these can be checked with exact theoretical results experimental observations. This process of verification allows us to predict new behaviours. In general, whenever one uses the MF approximation on the density matrix, one can obtain coupled equations in the expectation values of

observables, which can then be coded in a similar manner.

5 Acknowledgments

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